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DIMENSION, SUPERPOSITION OF FUNCTIONS AND SEPARATION OF POINTS, IN COMPACT METRIC SPACES

BY

Y. STERNFELD

Department of Mathematics, University of Haifa, Israel

ABSTRACT

It is proved that a compact metric space X is n-dimensional ($n \geqq 2$) if and only if there exist $2n+1$ functions $\varphi_1, \varphi_2, \dots, \varphi_{2n+1}$ in $C(X)$ so that each $f \in C(X)$ is representable as

$$f(x) = \sum_{i=1}^{2n+1} g_i(\varphi_i(x)) \qquad \text{with } g_i \in C(R), \quad 1 \leqq i \leqq 2n+1.$$

Equivalently, it is shown that $\dim X = n$ if and only if $C(X)$ is the algebraic sum of $2n+1$ subalgebras, each of which is isomorphic to $C(0,1)$. The properties of families $\{\varphi_i\}_{i=1}^{2n+1}$ which satisfy the above are studied, and they are characterized in terms of their ability to separate the points of X in some strong sense.

§1. Introduction

By a classical result of Menger and Nöbeling, every separable metric space of topological dimension n can be imbedded in the $(2n+1)$-dimensional Euclidean space R^{2n+1}. However, the fact that a given space X imbeds into R^{2n+1} does not determine the dimension of X. In this article we study a special type of imbeddings, which characterize the dimension of compact metric spaces.

Our starting point is the well-known superposition theorem of Kolmogorov [4]. It says that for $X = I^n$ ($n \geqq 2$) there exist $2n+1$ functions $\{\varphi_i\}_{i=1}^{2n+1} \subset C(X)$ of the form

$$(1.1) \qquad \varphi_i(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} \varphi_{i,j}(x_j), \qquad \varphi_{i,j} \in C(I), \quad 1 \leqq i \leqq 2n+1, \quad 1 \leqq j \leqq n$$

such that each $f \in C(X)$ admits a representation

$$(1.2) \qquad f(x) = \sum_{i=1}^{2n+1} g_i(\varphi_i(x)), \qquad x = (x_1, x_2, \dots, x_n) \in X, \quad g_i \in C(R).$$

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(I^n ($n \geqq 1$) is the n-cube $[0,1]^n$. Throughout this article X, Y will denote compact metric spaces, unless otherwise stated. $C(X)$ is the Banach space of real valued continuous functions on (X).) This remarkable theorem, which solved (negatively) Hilbert's 13-th problem, can be improved and generalized in several directions (see, e.g., [6] and [9]). We shall be interested in its extension to general compact metric spaces. To state it efficiently we introduce the following notation:

Definition 1.1. Let $F = \{\varphi_i\}_{i=1}^k$ be a family of continuous functions, $\varphi_i : X \to Y_i$, $1 \leqq i \leqq k$. F is said to be a *basic* family, if each $f \in C(X)$ admits a representation

$$(1.3) \qquad f(x) = \sum_{i=1}^{k} g_i(\varphi_i(x)), \qquad x \in X, \quad \text{with } g_i \in C(Y_i), \quad 1 \leqq i \leqq k.$$

Thus, the family of functions $\{\varphi_i\}_{i=1}^{2n+1}$ in Kolmogorov's theorem is a basic family. Note that it has the additional structure (1.1), but even if (1.1) is ignored Kolmogorov's theorem remains highly nontrivial. Given a compact metric space X, we shall be interested in basic families F on X, with $F \subset C(X)$. If $\dim X = n$ ($\dim X$ is the topological dimension of X) then by applying the Menger–Nöbeling theorem, and then Kolmogorov's theorem, we obtain a basic family $F \subset C(X)$ with $|F| =$ cardinality of $F = 2(2n+1)+1 = 4n+3$.

Ostrand [7] improved this result. In particular he proved:

$$(1.4) \qquad \text{If } \dim X \leqq n \ (n \geqq 0) \text{ then there exists a basic family } F \subset C(X) \text{ with } |F| \leqq 2n+1.$$

It is clear that the number $2n+1$ in (1.4) is the best possible. (There are n-dimensional spaces which do not imbed in R^{2n}.) But it turns out that it is the optimal in a much stronger sense; it cannot be reduced for *any* n-dimensional space X, not even when $X = I^n$ (and, in particular, the number $2n+1$ in Kolmogorov's theorem cannot be reduced.) This is the main result of this article.

Theorem 1. *Let X be a compact metric space and n a positive integer. Then* $\dim X \leqq n$ *if and only if there exists a basic family $F \subset C(X)$ with* $|F| \leqq 2n+1$.

Theorem 1 can be interpreted in several ways. Let us examine some.

If $F = \{\varphi_i\}_{i=1}^k \subset C(X)$ is basic, then the mapping φ from X to R^k whose coordinates are the elements of F is an imbedding, which also satisfies the following: any $f \in C(\varphi(X))$ can be represented as

$$(1.5) \qquad f(t_1, t_2, \ldots, t_k) = \sum_{j=1}^{k} g_i(t_i), \qquad (t_1, t_2, \ldots, t_k) \in \varphi(X), \quad g_i \in C(R).$$

Thus, Theorem 1 is equivalent to the following:

THEOREM 2. $\dim X \leqq n$ $(n \geqq 1)$ *if and only if there exists an imbedding* $\varphi : X \to R^{2n+1}$ *which satisfies* (1.5).

For a fixed $\varphi \in C(X)$ the collection

$$A = \{g(\varphi(x)) : g \in C(R)\} \tag{1.6}$$

is a closed subalgebra of $C(X)$, which contains the constant functions and is generated by one element. Conversely, every subalgebra A of $C(X)$ with the above-mentioned properties is of the form (1.6), with some $\varphi \in C(X)$. Hence, the following theorem is equivalent to Theorem 1.

THEOREM 3. $\dim X \leqq n$ $(n \geqq 1)$ *if and only if* $C(X)$ *is the (algebraic) sum of* $2n+1$ *subalgebras, each of which contains the constants and is generated by one element.*

Theorem 3 characterizes $\dim X$ in terms of the algebra structure of $C(X)$. We wish to mention some additional facts concerning this matter:

In [12], the following extension of (1.4) has been proved.

(1.7) Let $\dim X \leqq n$ $(n \geqq 0)$. There exist n spaces Y_j, with $\dim Y_j = 1$, $1 \leqq j \leqq n$, continuous mappings $\psi_j : X \to Y_j$, $1 \leqq j \leqq n$, and $2n+1$ functions $\{\varphi_i\}_{i=1}^{2n+1} \subset C(X)$, such that for every $0 \leqq k \leqq n$, every collection of k of the ψ_j's and $2(n-k)+1$ of the φ_i's forms a basic family.

Moreover, the Y_j's and ψ_j's can be so chosen that, with the exception of a set of first category in $C(X)^{2n+1}$, any $(2n+1)$-tuple $\{\varphi_i\}_{i=1}^{2n+1}$ of elements in $C(X)$ will satisfy the above.

Note that (1.4) follows from (1.7) by taking $k = 0$. In [12] it has been proved that the numbers k and $2(n-k)+1$ in (1.7) are the best possible for $n \leqq 6$. By the results of this article, the restriction $n \leqq 6$ in [12] can be removed. Let $\{\psi_j\}_{j=1}^{n}$ and $\{\varphi_i\}_{i=1}^{2n+1}$ be as in (1.7), and consider the following subalgebras of $C(X)$:

$$\begin{aligned} A_i &= \{g(\varphi_i(x)) : g \in C(R)\}, \qquad 1 \leqq i \leqq 2n+1; \\ B_j &= \{h(\psi_j(x)) : h \in C(Y_j)\}, \qquad 1 \leqq j \leqq n. \end{aligned} \tag{1.8}$$

(1.7) says that the sum of any k of the B_j's and any $2(n-k)+1$ of the A_i's is $C(X)$. Clearly, both the A_i's and the B_j's contain the constant functions, and the A_i's are generated by one element. The B_j's need not be generated by one element. Still, applying the following theorem of Katetov [3], the B_j's can be characterized in terms of their generators.

A subalgebra B of $C(X)$ is called analytic if B is closed, if $1 \in B$, and if for $f \in C(X)$, $f^2 \in B$ implies that $f \in B$.

A family $V \subset C(X)$ is an analytic generator of $C(X)$, if the smallest analytic subalgebra of $C(X)$ which contains V is $C(X)$. The analytic dimension of $C(X)$ is the smallest cardinality of an analytic generator. Katetov proved

$$\dim X = \text{analytic dimension of } C(X). \tag{1.9}$$

(For example: if $\Delta = \{0,1\}^N$ is the Cantor set, then the analytic dimension of $C(\Delta)$ is 0, i.e., the only analytic subalgebra of $C(\Delta)$ is $C(\Delta)$ itself. If T denotes the circle, then the analytic dimension of $C(T)$ is one, since $V = \{\sin t\}$ is an analytic generator. The reader may easily verify these facts.)

From (1.7), (1.8), and (1.9) it follows that the B_j's in (1.8) are analytically generated by one element. Hence the following stronger version of Theorem 3 holds.

THEOREM 4. $\dim X \leqq n$ $(n \geqq 1)$ *if and only if there exist subalgebras* A_i, $1 \leqq i \leqq 2n+1$, *and* B_j, $1 \leqq j \leqq n$, *of* $C(X)$, *which contain the constants, such that the* A_i*'s are generated by one element, and the* B_j*'s are analytically generated by one element, so that for each* $0 \leqq k \leqq n$, *the algebraic sum of any* k *of the* B_j*'s and any* $2(n-k)+1$ *of the* A_i*'s is* $C(X)$.

Obviously, a basic family separates the points of X, and simple examples show that the converse statement is false. It is therefore natural to study the stronger separation properties that basic families must share. A simple duality argument reveals those properties. This duality approach turns out to be highly significant. It exposes the real nature of basic families on one hand, and provides us with the main tool for the proof of Theorem 1 on the other.

Let $F = \{\varphi_i\}_{i=1}^k$ be a family of continuous functions on X, $\varphi_i : X \to Y_i$, $1 \leqq i \leqq k$. Let $Y = \bigcup_{i=1}^k Y_i$ denote the disjoint union of the Y_i's. Consider the bounded linear operator $T : C(Y) \to C(X)$ defined by

$$T(g_1, g_2, \dots, g_k)(x) = \sum_{i=1}^k g_i(\varphi_i(x)), \qquad x \in X, \quad (g_1, g_2, \dots, g_k) \in C(Y)$$
$$(\text{i.e., } g_i \in C(Y_i),\ 1 \leqq i \leqq k).$$

Clearly, F is basic if and only if T maps $C(Y)$ onto $C(X)$. This occurs if and only if T^* is an isomorphism, i.e., if and only if there exists some constant $\gamma > 0$ such that $\|T^*\mu\| \geqq \gamma \|\mu\|$ for all $\mu \in C(X)^*$. (Consult [1] for unexplained notation and facts concerning the duality argument.)

A routine check shows that for a Borel measure $\mu \in C(X)^*$, $T^*\mu = \sum_{i=1}^k \mu \circ \varphi_i^{-1}$, where $\mu \circ \varphi_i^{-1}$ is the measure of Y_i defined by $\mu \circ \varphi_i^{-1}(u) = \mu(\varphi_i^{-1}(u))$, $u \subset Y_i$ a Borel set. Thus

(1.10) F is basic if and only if there exists a constant $0 < \lambda \leqq 1$, such that for every $\mu \in C(X)^*$, $\|\mu \circ \varphi^{-1}\| \geqq \lambda \|\mu\|$ holds for some $\varphi \in F$.

Let us consider now families F which satisfy the conclusion of (1.10) for measures $\mu \in C(X)^*$, with a finite support (i.e., $\mu = \sum_{j=1}^m a_j \delta_{x_j}$, where δ_x, $x \in X$, is the Dirac measure with mass 1 at x, and $a_j \in R$; note that $\mu \circ \varphi^{-1} = \sum_{j=1}^m a_j \delta_{\varphi(x_j)}$.

DEFINITION 1.2. Let X and $\{Y_i\}_{i=1}^k$ be sets, and let $\varphi_i : X \to Y_i$ be functions. $F = \{\varphi_i\}_{i=1}^k$ is said to be a *uniformly separating family* (u.s.f. in short) if there exists a constant $0 < \lambda \leqq 1$ such that for each $\mu \in l_1(X)$, $\|\mu \circ \varphi\| \geqq \lambda \|\mu\|$ holds for some $\varphi \in F$.

REMARKS. It is easy to check that if in Definition 1.2 we replace "each $\mu \in l_1(X)$" by "each $\mu \in l_1(X)$ with a finite support" (i.e., $\mu = \sum_{j=1}^m a_j \delta_{x_j}$, $a_j \in R$), or even by "each $\mu = \sum_{j=1}^m a_j \delta_{x_j} \in l_1(X)$ with a_j an integer, and $\mu(X) = 0$" we still get an equivalent definition. By applying a duality argument similar to the one used above (cf. [10]), one can show that

(1.11) F is a u.s.f. if and only if each $f \in l_\infty(X)$ admits a representation

$$f(x) = \sum_{i=1}^k g_i(\varphi_i(x)), \qquad x \in X, \quad g_i \in l_\infty(Y_i)$$

(where $l_\infty(X)$ is the Banach space of bounded real valued functions on X).

Note that a u.s.f. F on X also satisfies the following: for any two disjoint finite subsets A and B of X, there exists some $\varphi \in F$ so that

$$|\varphi(A) \cap \varphi(B)| \leqq \tfrac{1}{2}(1-\lambda)(|A| + |B|),$$

i.e., if $A \cap B = \varnothing$ then $\varphi(A) \cap \varphi(B)$ is uniformly small. It was this property which motivated the choice of the terminology "uniformly separating family."

Thus, a basic family is a u.s.f. We do not know whether the converse statement (when applied to a family of continuous functions on a compact metric space) is true in general. (If F consists of at most two functions then it is true; cf. [10].)

We present some examples to illustrate this concept. In the first four X is a subset of I^2, while F consists of the two functions $\varphi_1(x, y) = x$, $\varphi_2(x, y) = y$.

EXAMPLE 1. X is the boundary of a rectangle with sides parallel to the axes (e.g. $X^*\{0,1\}\times I \cup I\times\{0,1\}$). Then F is not a u.s.f. on X (e.g., for $\mu = \delta_{(0,0)} + \delta_{(1,1)} - \delta_{(1,0)} - \delta_{(0,1)}$, $\mu\circ\varphi_i^{-1} = 0$ for $i = 1,2$).

EXAMPLE 2. X is the triangle with vertices at $(0,0)$, $(\frac{1}{2},0)$, and $(1,1)$. The reader may easily verify that for all $0 \neq \mu \in C(X)^*$, either $\mu\circ\varphi_1^{-1} \neq 0$ or $\mu\circ\varphi_2^{-1}\neq 0$, but still F is not a u.s.f. on X. (Actually F is not a u.s.f. on any closed curve in I^2; cf. [10].)

EXAMPLE 3. $X = \{\frac{1}{2}\}\times I \cup I \times\{\frac{1}{2}\}\setminus\{(\frac{1}{2},\frac{1}{2})\}$. F is a u.s.f. on X with $\lambda = \frac{1}{2}$.

EXAMPLE 4. $X = \{\frac{1}{2}\}\times I\cup I\times\{\frac{1}{2}\}$. f is still a u.s.f. on X with $\lambda = \frac{1}{3}$.

EXAMPLE 5. Let X denote the circle. Let $\{A_i\}_{i=1}^3$ be three disjoint arcs in X, and let $B_i = X\setminus A_i$ denote the complementary arcs. Let $F = \{\varphi_i\}_{i=1}^3 \subset C(X)$ be any family such that φ_i is one-to-one on B_i, $1 \leqq i \leqq 3$. Then F is basic on X with $\lambda = \frac{1}{3}$ (cf. [11]).

Note that in all the examples $\dim X = 1$. Examples in higher dimensions are much more complicated.

The following theorem, when combined with (1.4), provides a stronger version of Theorem 1.

THEOREM 5. *If* $\dim X = n \geqq 2$, *and* $F \subset C(X)$ *is a u.s.f., then* $|F| \geqq 2n+1$.

We prove Theorem 5 in the next section. There we shall formulate two theorems, and show how Theorem 5 follows from them. Both theorems, besides their role in the proof, will provide us with information on the structure of u.s.f. in general. The theorems will be proved in subsequent sections.

Note that the cases $n = 2,3,4$, of Theorem 5 have been proved in [10]. However, the proof presented there cannot be pushed through to larger values of n. We shall comment on this point again in Section 3.

Finally, we remark that our proof of Theorem 5 (or Theorem 1) cannot be shortened or simplified by narrowing the class of spaces to which it applies. Actually, the proof of Theorem 5 for the single space $X = I^n$ (which is the most interesting and important case) requires the same machinery and dimension theoretic arguments as the proof of the general case.

§2. Proof of Theorem 5

For a compact metric space X set

$$\alpha(X) = \min\{|F| : F\subset C(X), F \text{ a u.s.f.}\} \tag{2.1}$$

and for $n \geqq 0$ define

$$\alpha_n = \min\{\alpha(X) : \dim X = n\}. \tag{2.2}$$

Thus, by (1.4), $\alpha_n \leqq 2n+1$ for $n \geqq 0$; and Theorem 5 claims that for $n \geqq 2$, $\alpha_n = 2n+1$. (Obviously $\alpha_1 = 1$.) Let us first remark that

$$\text{for } n \geqq 2, \quad \alpha_{n+1} > \alpha_n \geqq n+1. \tag{2.3}$$

PROOF. Fix some $n \geqq 2$, and assume that $\alpha_n < n+1$. Thus, there exists some X with $\dim X = n$, and $F = \{\varphi_i\}_{i=1}^{\alpha_n} \subset C(X)$ a u.s.f. The mapping $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{\alpha_n}) : X \to R^{\alpha_n}$ is then an imbedding, and hence (as $\dim X = n$) we must have $\alpha_n = n$, and also, since a subset of R^n is n-dimensional if and only if it has a nonempty interior (cf. [2]) the interior of $\varphi(X)$ in R^n is nonempty. It follows that $\varphi(X)$ contains some n-cube; and to save notation, we may assume without loss of generality that $[-1,1]^n \subset \varphi(X)$. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, where $\varepsilon_j \in \{\pm 1\}$, denote the vertices of $[-1,1]^n$. For each such ε, let $s(\varepsilon) = \Pi_{j=1}^n \varepsilon_j$. Set also $x_\varepsilon = \varphi^{-1}(\varepsilon)$, and let $\mu \in l_1(X)$ be defined by $\mu = \Sigma_\varepsilon s(\varepsilon)\delta_{x_\varepsilon}$. Then $\|\mu\| = 2^n$, and it is easy to verify that $\mu \circ \varphi_i^{-1} = 0$ for all $1 \leqq i \leqq n$, which contradicts the assumption that F is a u.s.f. Hence $\alpha_n \geqq n+1$.

Assume now that for some $n \geqq 2$, $\alpha_{n+1} = \alpha_n$. It follows that there exist some X with $\dim X = n+1$, and $F = \{\varphi_i\}_{i=1}^{\alpha_n} \subset C(X)$ a u.s.f. The function φ_1 maps X into R and hence (see (3.2) in §3 of this article) there exists some $t \in R$ such that $\dim \varphi_1^{-1}(t) \geqq n$. Obviously, $F' = \{\varphi_i\}_{i=2}^{\alpha_n}$ is then a u.s.f. on $\varphi_1^{-1}(t)$, and $|F'| = \alpha_n - 1$ which contradicts the definition of α_n.

Our proof that for $n \geqq 2$, $\alpha_n = 2n+1$, consists of two major steps. Both of these steps reveal some pattern of u.s.f. in general. To gain some intuition towards the first step, consider the space $X = I^2$, and a basic family $F \subset C(X)$, which consists of continuously differentiable functions, and which is minimal in the sense that no $F' \subsetneqq F$ is basic on any $X' \subset X$ with nonempty interior. From elementary calculus it then follows that every pair of elements of F, when regarded as a mapping from X into R^2, maps X to a subset of R^2 with a nonempty interior. This is no longer true without the differentiability assumption. In the first step we prove that this is still the case with "many" of the n-tuples of elements of F. To prove that $\alpha_n = 2n$ is impossible, we have to show that, given $F = \{\varphi_i\}_{i=1}^{2n} \subset C(X)$ where $\dim X = n$, there exists $\mu \in l_1(X)$ with $\|\mu \circ \varphi_i^{-1}\|$ small with respect to $\|\mu\|$ for all $\varphi_i \in F$.

Apparently, the existence of a "Cartesian product structure" in "many" of the n-tuples of elements of F (i.e., the n-tuples which by the first step map X to a set with a nonempty interior in R^n) is useful when such $\mu \in l_1(X)$ are to be constructed. (A simple example which illustrates this fact is the proof that $\alpha_n \geqq n+1$ in (2.3).) In the second step we show that this is really the case. We shall introduce now the necessary terminology, and state two theorems: Theorem 6 (for the first step) and Theorem 7 (for the second). Then we shall deduce Theorems 5 from these theorems. Theorems 6 and 7 will be proved in the following sections.

DEFINITION 2.1. Let $n \geqq 1$ be an integer, let $\beta = \{\beta_i\}_{i=1}^n$ be a strictly increasing sequence of positive integers, and let K be a finite set. The concept of a tree T of order n and type β of subsets of K will be defined by induction on n.

T is a tree of order 1 and type $\beta = \{\beta_1\}$ of subsets of K, if there exists a subset T^* of K, with $|T^*| \geqq \beta_1$ such that $T = \{\{i\} : i \in T^*\}$ (i.e., T is a family of subsets of K, of cardinality one each, and T contains at least β_1 elements).

Assume that a tree of order r and type β of subsets of K has been defined for $1 \leqq r \leqq n-1$. T is a tree of order n and type $\beta = \{\beta_1, \dots, \beta_n\}$ of subsets of K, if there exists a subset $T^* \subset K$, with $|T^*| \geqq \beta_n$, such that to each $i \in T^*$, there corresponds a tree T_i of order $n-1$ and type $\{\beta_1, \dots, \beta_{n-1}\}$ of subsets of $T^* \setminus \{i\}$, and $T = \{\{i\} \cup a : a \in T_i, i \in T^*\}$.

Note that a tree T of order n and type β of subsets of K is a family of subsets of K (actually of T^*), of cardinality n each. One can look upon the elements of T as "branches" of a tree, which has the elements of T^* in its basis, each $i \in T^*$ branches to at least β_{n-1} elements of $T^* \setminus \{i\}$, each such element j branches to at least β_{n-2} elements of $T_i^* \setminus \{j\}$, and so on. (The branches are considered here as sets — not ordered sets — and hence different branches may define the same element of T.)

DEFINITION 2.2. Let X and Y be topological spaces, and let $f : X \to Y$ be continuous. f is an *interior* function, if for each nonempty open subset U of X, $f(U)$ has a nonempty interior in Y.

DEFINITION 2.3. Let X and Y_i, $1 \leqq i \leqq k$, be sets and $\varphi_i : X \to Y_i$ functions. For a subset a of $\{1, 2, \dots, k\}$ let $\varphi_a : X \to \Pi_{i \in a} Y_i$ be defined by: $(\varphi_a(x))_i = \varphi_i(x)$, $x \in X$, $i \in a$.

THEOREM 6. *Let X be an n-dimensional compact metric space ($n \geqq 2$) and let $\{\varphi_i\}_{i=1}^k \subset C(X)$ be a u.s.f. on X.*

Then there exists an n-dimensional closed subset X' of X and a tree T of order n and type $\{2, \alpha_2, \alpha_3, \dots, \alpha_n\}$ of subsets of $\{1, 2, \dots, k\}$, such that for all $a \in T$, $\varphi_a : X' \to R^n$ is interior.

REMARK. Let us call a tree which satisfies the conclusion of Theorem 6, *an interior tree with respect to* $F = \{\varphi_i\}_{i=1}^k$. Since in Theorem 5 we shall prove that $\alpha_n = 2n + 1$ for $n \geqq 2$, it follows from Theorem 6 that each real valued u.s.f. F on an n-dimensional compact metric space admits an interior tree T of order n and type $\{2, 5, 7, \ldots, 2n-1, 2n+1\}$.

This result can be slightly improved; it turns out that each such u.s.f. admits an interior tree of order n and type $\{3, 5, 7, \ldots, 2n+1\} = \{2l+1\}_{l=1}^n$. The proof of this fact requires more delicate arguments than the arguments needed for the proof of Theorem 6, and since we do not need it, we shall not present it here. (We refer to the proof of Theorem 5 (case (ii)) of [12], in which the additional arguments which are needed in order to obtain a tree of type $\{3, 5, \ldots, 2n+1\}$ are presented.)

The type $\{3, 5, \ldots, 2n+1\}$ cannot be improved. Indeed, let $X = I^2$. Then $\dim X = 2$, and by (1.7), there exists a u.s.f. F on X, $F = \{\varphi_1, \varphi_2, \varphi_3, \psi\}$, with $\varphi_i \in C(X)$, $i = 1, 2, 3$, and $\psi : X \to Y$, where $\dim Y = 1$. Hence there exists a u.s.f. $\{\psi_i\}_{i=1}^3 \subset C(Y)$ on Y so that for all $a \subset \{1, 2, 3\}$ $(a \neq \emptyset)$, $\dim \psi_a(Y) = 1$. (To see this one needs the stronger version of (1.7), i.e., that up to a set of first category, all triples of elements of $C(Y)$ form a u.s.f.) Let $\tau_i \in C(X)$ be defined by $\tau_i(x) = \psi_i(\psi(x))$, $i = 1, 2, 3$. One checks easily that $F' = \{\varphi_1, \varphi_2, \varphi_3\} \cup \{\tau_1, \tau_2, \tau_3\}$ is a u.s.f. on X. Moreover, F' does not admit an interior tree of type $\{4, 5\}$ (and order 2). This follows from the fact that for $a = \{i, j\}$, $1 \leqq i < j \leqq 3$, $\tau_a(X) = \psi_a(Y)$, and hence $\dim \tau_a(X) = \dim \psi_a(Y) = 1$, i.e., the interior of $\tau_a(X)$ in R^2 is empty.

We turn now to the second step. First we introduce the concept of an array.

DEFINITION 2.4. Let X and Y_i, $i \in T^*$ be sets, and let $F = \{\varphi_i\}_{i \in T^*}$ be a family of functions, $\varphi_i : X \to Y_i$, where T^* is a finite set of indices. Let n be a positive integer, and $c > 0$.

A measure $\mu \in l_1(X)$ is said to be an array of order n and constant c, with respect to F, if the following holds:

(ar.1) μ can be represented as $\mu = \sum_{j=1}^m \varepsilon(j)\delta_{x_j}$, where $\varepsilon(j) \in \{\pm 1\}$ and $\{x_j\}_{j=1}^m \subset X$ is a finite sequence.

(ar.2) $\|\mu\| = m$.

(ar.3) For each $i \in T^*$, there exists a subset L_i of $[m] = \{1, 2, \ldots, m\}$, so that:

(ar.3.1) $\mu_i = \sum_{j \in L_i} \varepsilon(j)\delta_{x_j}$ satisfies $\mu_i \circ \varphi_i^{-1} = 0$, $i \in T^*$.

(ar.3.2) If for $j \in [m]$ we set

$$\sigma(j) = \{i : i \in T^*, j \in L_i\}$$

then $|\sigma(j)| \leqq 2n$ and

$$|\{j : j \in [m], |\sigma(j)| = 2n\}| \geqq \|\mu\| - c\|\mu\|^{(n-1)/n}.$$

Note that (ar.2) is equivalent to:

(ar.2′) If $x_{j_1} = x_{j_2}$ then $\varepsilon(j_1) = \varepsilon(j_2)$

and also that (ar.3.1) is equivalent to:

(ar.3.1′) There exists a decomposition E_i of L_i into disjoint pairs $E_i = \{\{j, j'\}\}$ such that, for $\{j, j'\} \in E_i$, $\varepsilon(j) \cdot \varepsilon(j') = -1$ and $\varphi_i(x_j) = \varphi_i(x_{j'})$ hold.

The verification of these facts are left to the reader. The usefulness of arrays to our goal is reflected in the following proposition:

Proposition 2.1. *Let μ be an array of order n and constant c w.r.t. $F = \{\varphi_i\}_{i \in T^*}$. If $|T^*| = 2n$, then for all $i \in T^*$, $\|\mu \circ \varphi_i^{-1}\| / \|\mu\| \leqq c\|\mu\|^{-1/n}$.*

Proof. If $|T^*| = 2n$ then by (ar.3.2)

$$|\{j : j \in [m], \sigma(j) = T^*\}| \geqq \|\mu\| - c\|\mu\|^{(n-1)/n},$$

and also, if $\sigma(j) = T^*$, then $j \in L_i$ for all $i \in T^*$; so, in particular, for all $i \in T^*$, $L_i \supset \{j : \sigma(j) = T^*\}$ and thus $|L_i| \geqq \|\mu\| - c\|\mu\|^{(n-1)/n}$. Note also that, by (ar.2),

$$\|\mu_i\| = \Big\|\sum_{j \in L_i} \varepsilon(j)\delta_{x_j}\Big\| = |L_i| \geqq \|\mu\| - c\|\mu\|^{(n-1)/n},$$

and

$$\|\mu - \mu_i\| = \Big\|\sum_{j \in [m] \setminus L_i} \varepsilon(j)\delta_{x_j}\Big\| \leqq c\|\mu\|^{(n-1)/n}.$$

Hence by (ar.3.1)

$$\|\mu \circ \varphi_i^{-1}\| = \|(\mu - \mu_i) \circ \varphi_i^{-1} + \mu_i \circ \varphi_i^{-1}\| = \|(\mu - \mu_i) \circ \varphi_i^{-1}\| \leqq \|\mu - \mu_i\| \leqq c\|\mu\|^{(n-1)/n}$$

and the proposition follows.

The following theorem provides sufficient conditions for the existence of arrays.

Theorem 7. *Let T be a tree of order n ($n \geqq 2$) and type $\{2, 4, 6, \ldots, 2n\}$. Let X*

be a topological space, and let Y_i, $i \in T^*$, be topological spaces in which each nonempty open set contains two disjoint nonempty open sets. Let $F = \{\varphi_i\}_{i \in T^*}$ be a family of continuous functions, $\varphi_i : X \to Y_i$, so that for all $a \in T$, $\varphi_a : X \to \Pi_{i \in a} Y_i$ is interior.

Then there exists a constant $c = c(n, |T^*|)$ (which depends only on n and $|T^*|$) such that for every integer $L \geqq 1$, there exists in X an array μ of order n, norm L and constant c, w.r.t. F.

We proceed now to the proof of Theorem 5. We shall show, by induction on $n \geqq 2$, that $\alpha_n \geqq 2n+1$. Recall that by (2.3), $\alpha_{n+1} > \alpha_n \geqq n+1$, and that $\alpha_n \leqq 2n+1$.

PROOF OF THEOREM 5. *The case $n = 2$.* Let us see first that $\alpha_2 \geqq 4$. If not, then there exists a two-dimensional compact metric space X, and a u.s.f. $F = \{\varphi_i\}_{i=1}^3 \subset C(X)$. Hence, by Theorem 6, there exists a 2-dimensional compact subset X' of X, and an interior tree of order 2 and type $\{2,3\}$ w.r.t. F. Thus, for all $a \subset \{1,2,3\}$ with $|a| = 2$, $\varphi_a : X' \to R^2$ is interior. Set $\varphi_4 = \varphi_3$. Then $F = \{\varphi_i\}_{i=1}^4$, and one checks easily that $T = \{\{1,2\},\{1,3\},\{2,3\},\{4,1\},\{4,2\}\}$ is a tree of order 2, and type $\{2,4\}$, with $|T^*| = 4$, w.r.t. which $F = \{\varphi_i\}_{i=1}^4$ is interior (on X').

From Theorem 7 it now follows that there exists a constant c such that X' contains an array μ, of order 2, of arbitrary norm k, and constant c (independent of k) w.r.t. F. From Proposition 2.1, it follows that $\|\mu \circ \varphi_i^{-1}\| / \|\mu\| \leqq ck^{-1/2}$, for all $i \in T^* = \{1,2,3,4\}$, i.e., F is not a u.s.f. Hence $\alpha_2 \geqq 4$.

Assume that $\alpha_2 = 4$. Then, again, let $F = \{\varphi_i\}_{i=1}^4 \subset C(X)$ be a u.s.f. on some 2-dimensional compact metric space X. By Theorem 6, there exists a tree T, of order 2 and type $\{2,4\}$, of subsets of $\{1,2,3,4\}$ which is interior w.r.t. F on some $X' \subset X$, and clearly $|T^*| = 4$. Applying Theorem 7, and Proposition 2.1 once again, we obtain a contradiction. Hence $\alpha_2 = 5$.

Assume now that $\alpha_r = 2r+1$ for $2 \leqq r \leqq n-1$. Then $2n+1 \geqq \alpha_n > \alpha_{n-1} = 2(n-1)+1 = 2n-1$, i.e., $\alpha_n \geqq 2n$, and we have to show that $\alpha_n = 2n+1$. So, assume $\alpha_n = 2n$, and let X be an n-dimensional compact metric space, with $F = \{\varphi_i\}_{i=1}^{2n} \subset C(X)$ a u.s.f.

By Theorem 6, there exists an n-dimensional subset X' of X, and a tree T of order n and type $\{2,5,7,\ldots,2n-1,2n\}$ of subsets of $\{1,2,\ldots,2n\}$, which is interior w.r.t. F on X'. Clearly, T is also of type $\{2,4,6,\ldots,2n-2,2n\}$, and $|T^*| = 2n$.

Applying Theorem 7, and Proposition 2.1, we obtain a measure μ on X with $\|\mu \circ \varphi_i^{-1}\| / \|\mu\| \leqq ck^{-1/n}$ for all $1 \leqq i \leqq 2n$ where k is arbitrary, and c independent of k, which contradicts the assumption that F is a u.s.f.

§3. Proof of Theorem 6

We shall first prove the following weaker version of Theorem 6.

THEOREM 6′. *Let $F=\{\varphi_i\}_{i=1}^k \subset C(X)$ be a u.s.f. on an n-dimensional compact metric space X $(n \geqq 2)$.*

Then there exists a tree T of order n and type $\{2,\alpha_2,\alpha_3,\dots,\alpha_n\}$ of subsets of $\{1,2,\dots,k\}$ such that $\dim\varphi_a(X)=n$ for all $a\in T$.

PROOF. We shall use induction on $n\geqq 2$, and begin with the case $n=2$. So let $\dim X=2$ and let $F=\{\varphi_i\}_{i=1}^k\subset C(X)$ be a u.s.f. on X.

Note first that we may assume without loss of generality that F is a minimal u.s.f. on X in the following sense: no subfamily $F'\subsetneqq F$ is a u.s.f. on any closed 2-dimensional subset X' of X. Indeed, if $F'\subsetneqq F$ is a u.s.f. on some closed 2-dimensional $X'\subset X$, then we restrict ourselves to X' and F' instead of X and F; if there is still an $X''\subset X'$ closed $\dim X''=2$, and $F''\subsetneqq F'$ a u.s.f. on X'', then we pass to X'' and F''. As this procedure must obviously stop after a finite number of steps, we end up with a 2-dimensional compact subset W of X, and some $G\subset F$ which is a minimal u.s.f. on W. So, we shall assume that $X=W$ and $G=F$.

Reçall that an n-dimensional compact metric space X is called n-dimensional Cantor manifold, if for all $W\subset X$ closed with $\dim W\leqq n-2$, $X\setminus W$ is connected. By ([2], Th. VI.8, p. 94) each n-dimensional compact metric space contains some n-dimensional Cantor-manifold. In particular, our X contains a 2-dimensional Cantor-manifold, and hence we may assume without loss of generality that X itself is such.

Recall also that for a mapping $f:X\to Y$, $\dim f$ is defined by

$$\dim f=\sup\{\dim f^{-1}(y): y\in Y\}.$$

The following lemma, which will be proved at the end of this section, shows that under our assumptions, for all $a\subset\{1,2,\dots,k\}$ with $|a|=k-1$, $\dim\varphi_a=0$.

LEMMA 3.1. *Let X be an n-dimensional Cantor-manifold, and let $F=\{\varphi_i\}_{i=1}^k\subset C(X)$ be a minimal u.s.f. on X (i.e., no $F'\subsetneqq F$ is a u.s.f. on any closed n-dimensional subset of X).*

Then for each $a\subset\{1,2,\dots,k\}$ with $|a|=k-1$, $\dim\varphi_a=0$.

At this point we wish to recall some facts from dimension theory.

(3.2) *Hurewicz's theorem on mappings which lower dimension* ([2], Th.VI.7, p. 91). Let X and Y be separable metric spaces and let $f:X\to Y$ be a closed mapping. Then $\dim X\leqq\dim Y+\dim f$.

Recall that for a topological space X, $\operatorname{dc} X$ (the dimension of connectness of X) is defined by: $\operatorname{dc} X \geqq n$ if no closed subset $W \subset X$ with $\dim W \leqq n-2$ separates X (cf. [5], p. 164 for more details). Thus, e.g., X is an n-dimensional Cantor-manifold if X is compact and $\dim X = \operatorname{dc} X = n$.

The following follows easily from (3.2) (see [10], Th. 4.19, p. 76 for a proof).

(3.3) *Corollary of Hurewicz's Theorem.* Under the assumption of Hurewicz's Theorem $\operatorname{dc} X \leqq \operatorname{dc} Y + \dim f$.

Finally, we state the following

(3.4) *Theorem on dimension of projections.* Let $W \subset R^m$ be compact with $\operatorname{dc} W \geqq n$. If $\dim P_{\{i\}}(W) = 1$ for some $1 \leqq i \leqq m$, then there exists a subset b of $\{1, 2, \ldots, m\} \setminus \{i\}$ with $|b| = n-1$ such that $\dim P_{\{i\} \cup b}(W) = n$.

Here P_b denotes the canonical coordinate projection of R^m onto R^b, $b \subset \{1, 2, \ldots, m\}$.

(3.4) is proved in [10] (Th. 4.9, p. 74). Let us mention that the cases $n = 2, 3, 4$ of Theorem 5 are also proved in [10]. There, the author also conjectured an extension of (3.4) which could have been used to extend the proof of Theorem 5 in [10] to all $n \geqq 2$. However, Pixley [8] has shown that the extension of (3.4), suggested in [10], is false. The course of proof of Theorem 5 in this article, and in particular the notion of a tree, were introduced to bypass this obstacle.

We can now conclude the proof of the case $n = 2$ of our theorem. We shall show that, for each $1 \leqq i \leqq k$, there correspond to indices, j_1, j_2 in $\{1, 2, \ldots, k\} \setminus \{i\}$, such that $\dim \varphi_{\{i, j_1\}}(X) = \dim \varphi_{\{i, j_2\}}(X) = 2$. Note that if we accomplish this then we are done, since then the tree T with $T^* = \{1, 2, \ldots, k\}$, and $T_i = \{\{j_1\}, \{j_2\}\}$ for $i \in T^*$, is of order 2 and type $(2, \alpha_2)$ (obviously $k \geqq \alpha_2$), and for all $a \in T$ (*i.e.*, $a = \{i, j_1\}$ or $a = \{i, j_2\}$, $i \in T^*$) $\dim \varphi_a(X) = 2$.

So, let $1 \leqq i \leqq k$. Set $[k] = \{1, 2, \ldots, k\}$. Then as $F = \{\varphi_i\}_{i=1}^k$ is a u.s.f., $\varphi_{[k]} : X \to R^k$ is a homeomorphism, and thus $W = \varphi_{[k]}(X)$ is a 2-dimensional Cantor-manifold in R^k. By the minimality of F, $P_{\{i\}}(W) = \varphi_i(X)$ is a nondegenerate interval in R, and hence $\dim P_{\{i\}}(W) = 1$. By (3.4) there exists some $j_1 \in [k] \setminus \{i\}$ such that $\dim P_{\{i\} \cup \{j_1\}}(W) = 1$. (Note that $P_{\{i\} \cup \{j_1\}}(W) = \varphi_{\{i, j_1\}}(X)$.)

Set $a = [k] \setminus \{j_1\}$, and $V = \varphi_a(X) \subset R^{|a|}$, then $|a| = k-1$, and by Lemma 3.1, $\dim \varphi_a = 0$. Hence by (3.3) $\operatorname{dc} \varphi_a(X) \geqq \operatorname{dc} X - \dim \varphi_a = 2$. Thus, V is a compact subset of R^{k-1}, $\operatorname{dc} V \geqq 2$ and $\dim P_{\{i\}}(V) = \dim \varphi_i(X) = 1$. By (3.4) again, there exists some $j_2 \in a \setminus \{i\}$, such that

$$\dim P_{\{i\} \cup \{j_2\}}(V) = \dim \varphi_{\{i, j_2\}}(X) = 2$$

and the case $n = 2$ of the theorem follows.

Assume now that the theorem holds for $2 \leqq m \leqq n-1$, and let $F = \{\varphi_i\}_{i=1}^k \subset C(X)$ be a u.s.f. on an n-dimensional compact metric space X. An obbvious reduction (as in the case $n = 2$) allows us to assume that X is an n-dimensional Cantor-manifold, and that F is a minimal u.s.f. on X. Clearly $k \geqq \alpha_n$, and we shall construct a tree T of order n and type $(2, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n)$ of subsets of $[k]$, with $T^* = [k]$, such that for all $b \in T$, $\dim \varphi_b(X) = n$. To do this it suffices to show that, for each $1 \leqq i \leqq k$, there corresponds a tree T_i of order $n-1$ and type $(2, \alpha_2, \ldots, \alpha_{n-1})$ of subsets of $\{1, 2, \ldots, k\} \setminus \{i\}$ such that for each $a \in T_i$, $\dim \varphi_{\{i\} \cup a}(X) = n$. So let $1 \leqq i \leqq k$, and to save notation assume that $i = 1$. From the minimality of F it follows that $\varphi_1(X)$ is a closed interval $[\alpha, \beta]$ in R, with $\alpha < \beta$. For $\alpha < t < \beta$, t separates $[\alpha, \beta]$ and hence $\varphi_1^{-1}(t)$ separates X. The fact that X is a n-dimensional Cantor-manifold implies that $\dim \varphi_1^{-1}(t) \geqq n-1$, and from the minimality of F it follows that actually $\dim \varphi_1^{-1}(t) = n-1$ (since $\{\varphi_i\}_{i=2}^k$ is a u.s.f. on $\varphi_1^{-1}(t)$). Hence, by the induction hypothesis, there exists a tree $T_1(t)$, of order $n-1$ and type $(2, \alpha_2, \ldots, \alpha_{n-1})$, of subsets of $\{2, 3, \ldots, k\}$, such that for all $a \in T_1(t)$. $\dim \varphi_a(\varphi_1^{-1}(t)) = n-1$.

For a tree S of order $n-1$ and type $(2, \alpha_2, \ldots, \alpha_{n-1})$ of subsets of $\{2, 3, \ldots, k\}$, set

$$A_S = \{t : \alpha < t < \beta, \dim \varphi_a(\varphi_1^{-1}(t)) = n-1 \text{ for all } a \in S\}.$$

The above argument shows that $\bigcup_S A_S = \{t : \alpha < t < \beta\}$, since $t \in A_{T_1(t)}$. Since the number of such trees is finite, there exists some tree T_1 such that A_{T_1} is of second category in $[\alpha, \beta]$. (Note that A_{T_1} is not necessarily closed.)

We shall see now that for each $a \in T_1$, $\dim \varphi_{\{1\} \cup a}(X) = n$. So fix some $a \in T_1$. Recall that $|a| = n-1$. Let $\{B_l\}_{l=1}^\infty$ be a sequence of closed $(n-1)$-dimensional cubes in R^a, whose interiors form a basis for the topology of R^a.

Set

$$E_l = \{t : t \in A_{T_1}, B_l \subset \varphi_a(\varphi_1^{-1}(t))\}, \qquad l \geqq 1.$$

We claim that E_l is closed in R, and that $A_{T_1} \subset \bigcup_{l=1}^\infty E_l$. To see that E_l is closed, let $\{t_m\}_{m=1}^\infty \subset E_l$ be a sequence, so that $t_m \xrightarrow[m \to \infty]{} t_0$, $t_0 \in R$, and we shall see that $t_0 \in E_l$. Since X is compact and φ_1 is continuous, $\varphi_1^{-1} : [\alpha, \beta] \to 2^X$ is upper-semicontinuous. Hence $\overline{\lim}_m \varphi_1^{-1}(t_m) \subset \varphi_1^{-1}(t_0)$, and since $t_m \in E_l$ for $m \geqq 1$, $B_l \subset \varphi_a(\varphi_1^{-1}(t_m))$. Hence also

$$B_l \subset \overline{\lim_m} \, \varphi_a(\varphi_1^{-1}(t_m)) \subset \varphi_a(\varphi_1^{-1}(t_0)), \qquad \text{i.e., } t_0 \in E_l.$$

To see that $A_{T_1} \subset \bigcup_{l=1}^\infty E_l$, fix some $t \in A_{T_1}$. Then $\dim \varphi_a(\varphi_1^{-1}(t)) = n-1$, hence

$\varphi_a(\varphi_1^{-1}(t))$ has a nonempty interior in R^a, and thus $\varphi_a(\varphi_1^{-1}(t))$ contains some B_l, $l \geqq 1$, i.e., $t \in E_l$.

From the fact that A_{T_1} is of second category, it now follows that there exists some $l \geqq 1$ such that E_l has a non-empty interior in R, i.e., E_l contains some interval J. Then for all $t \in J$, $B_l \subset \varphi_a(\varphi_1^{-1}(t))$, i.e., $J \times B_l \subset \varphi_{\{1\}\cup a}(X)$, and since $J \times B_l$ is an n-cube, it follows that $\dim \varphi_{\{1\}\cup\{a\}}(X) = n$.

This proves Theorem 6′.

Proof of Theorem 6. Let $F = \{\varphi_i\}_{i=1}^k$ be a u.s.f. on an n-dimensional compact metric space X ($n \geqq 2$). Let $X' \subset X$ be an n-dimensional Cantor-manifold. By Theorem 6, there exists a tree T of order n and type $(2, \alpha_2, \dots, \alpha_n)$ of subsets of $\{1, 2, \dots, k\}$ such that for all $a \in T$, $\dim \varphi_a(X') = n$. If, for all $a \in T$, φ_a is interior on X', then we are done. If not, then there exist an open $\emptyset \neq U \subset X'$ and $a \in T$ such that $\varphi_a(U)$ has empty interior in R^a, i.e., $\dim \varphi_a(U) \leqq n-1$. Let $X'' \subset U$ be an n-dimensional Cantor-manifold. Another application of Theorem 6′ yields a further tree T'' (of the same order and type) so that $\dim \varphi_a(X'') = n$ for all $a \in T''$ (obviously $T'' \neq T$). If φ_a is interior on X'' for all $a \in T''$ then we are done. If not, there exists some $\emptyset \neq U \subset X''$ open and $a \in T''$ with $\dim \varphi_a(U) \leqq n-1$, and the above procedure can be continued. Since it must stop after finitely many steps, we shall end up with an n-dimensional Cantor-manifold $X^* \subset X$, and a tree T^* of order n and type $(2, \alpha_2, \dots, \alpha_n)$ so that for each $a \in T^*$, φ_a is interior on X^*. This proves Theorem 6.

For the proof of Lemma 3.1 we shall need the following lemma.

Lemma 3.2. *Let $F = \{\varphi_i\}_{i=1}^k$ be a u.s.f. on a set X. Let a, b be subsets of $[k]$ with $a \cup b = [k]$, and $a \cap b = \emptyset$. If φ_a is constant on some subset W of X, and $Z \subset \varphi_b^{-1}(\varphi_b(W)) \setminus W$, then $\{\varphi_i\}_{i \in a}$ is a u.s.f. on Z.*

Proof. Let $z \in Z$; then $\varphi_b(z) \in \varphi_b(W)$. Hence there exists some point $\tau(z) \in W$ such that $\varphi_b(z) = \varphi_b(\tau(z))$. Let now $\mu = \Sigma_j a_j \delta_{z_j} \in l_1(Z)$ be such that $\mu(Z) = 0$. Set $\mu' = \Sigma_j a_j \delta_{\tau(z_j)} \in l_1(W)$, and also $\tilde{\mu} = \mu - \mu'$. Then $\tilde{\mu} \circ \varphi_b^{-1} = 0$ (since $\varphi_b(z_j) = \varphi_b(\tau(z_j))$). Hence, since F is a u.s.f. on X, there must be some $i \in a$ such that $\|\tilde{\mu} \circ \varphi_i^{-1}\| \geqq \lambda \|\tilde{\mu}\|$. (Note that $\|\tilde{\mu}\| \geqq \|\mu\|$ since $Z \cap W = \emptyset$.) But for $i \in a$, φ_i is constant on W. Thus $\mu' \circ \varphi_i^{-1} = 0$. So

$$\tilde{\mu} \circ \varphi_i^{-1} = (\mu - \mu') \circ \varphi_i^{-1} = \mu \circ \varphi_i^{-1} - \mu' \circ \varphi_i^{-1} = \mu \circ \varphi_i^{-1}$$

and

$$\|\mu \circ \varphi_i^{-1}\| \geqq \lambda \|\tilde{\mu}\| \geqq \lambda \|\mu\|,$$

i.e., $\{\varphi_i\}_{i \in a}$ is a u.s.f. on Z.

PROOF OF LEMMA 3.1. Let $a \subset [k]$ with $|a| = k-1$ be given. To save notation assume that $a = \{1, 2, \dots, k-1\}$. Assume also that φ_a is not 0-dimensional. Then there exists some $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{k-1}) \in R^{k-1}$ such that $\dim \varphi_a^{-1}(\alpha) \geqq 1$. Set $W = \varphi_a^{-1}(\alpha)$. Then φ_a is constant on W, and thus, by Lemma 3.2, $\{\varphi_i\}_{i=1}^{k-1}$ is a u.s.f. on $Z = \varphi_k^{-1}(\varphi_k(W)) \setminus W$. But φ_k is a homeomorphism on W, so $\dim \varphi_k(W) = 1$ and $\varphi_k(W)$ must contain some open interval $J \subset R$. Hence $\varphi_1^{-1}(J)$ is an open subset of X which must contain some closed n-dimensional subset X' of X, which is contained in Z, and this contradicts the minimality of F.

§4. Proof of Theorem 7

In order to prove Theorem 7, we state and prove a stronger result. Let us first introduce some conventions. Throughout this section "an open set" will always mean "a nonempty open set". $W \subset\subset X$ denotes "W is an open subset of X". We also assume that the topological spaces Y considered in this section enjoy the following property: for any $U \subset\subset Y$, there exist $W \subset\subset U$ and $V \subset\subset U$ with $Q \cap V = \emptyset$. If $\varphi : X \to Y$ is a function, and $\alpha = \{\alpha_i\}_{i=1}^m \subset X$ and $\beta = \{\beta_j\}_{j=1}^k \subset Y$ are finite sequences, then by "$\varphi(\alpha) = \beta$" we shall understand that $m = k$ and that there exists a permutation of $\{1, 2, \dots, k\}$ such that $\varphi(\alpha_i) = \beta_{\pi(i)}$, $1 \leqq i \leqq k$.

DEFINITION 4.1. Let $\mu = \sum_{j=1}^L \varepsilon(j)\delta_{x_j}$ be an array, of order n and with constant c, w.r.t. some family $F = \{\varphi_i\}_{i \in T^*}$ of functions on a set X. We say that μ is *a nomral array* if, in addition to (ar.1), (ar.2), (ar.3), (ar.3.1) and (ar.3.2), μ also satisfies

$$\text{(ar. } N) \qquad \text{for every } \sigma \subset T^*, \quad \Big| \sum_{j:\sigma(j)=\sigma} \varepsilon(j) \Big| \leqq cL^{(n-1)/n}.$$

THEOREM 8. *Let X, and Y_i, $1 \leqq i \leqq k$, be topological spaces, and let $\{\varphi_i\}_{i=1}^k$ be continuous functions, $\varphi_i : X \to Y_i$, $1 \leqq i \leqq k$. Let b be a subset of $\{1, 2, \dots, k\}$, and let T be a tree of order n $(n \geqq 1)$ and type $\{2, 4, 6, \dots, 2n\}$ of subsets of $\{1, 2, \dots, k\} \setminus b$, so that $|b| + n \geqq 2$, and such that for all $a \in T$, $\varphi_{a \cup b}$ is interior.*

Then there exists a constant $c = c(n, |T^*|)$ so that the following holds: For every integer $L \geqq 1$, and every $U \subset\subset X$, there exists some $V \subset\subset U$ such that, given any two disjoint sequences $\beta^+ = \{\beta_j^+\}_{j=1}^{L^+}$ and $\beta^- = \{\beta_j^-\}_{j=1}^{L^-}$ in $\varphi_b(V)$, with $|L^+ - L^-| \leqq 1$ and $L^+ + L^- = L$, there exists in U a normal array $\mu = \sum_{j=1}^L \varepsilon(j)\delta_{x_j}$, of order n, constant c, and norm L, w.r.t. $F = \{\varphi_i\}_{i \in T^*}$, so that $\varphi_b(\{x_j\}_{\varepsilon(j)=1}) = \beta^+$ and $\varphi_b(\{x_j\}_{\varepsilon(j)=-1}) = \beta^-$.

Note that Theorem 7 can be obtained from Theorem 8 by taking $n \geqq 2$, $b = \emptyset$, and $U = X$. The array μ constructed in this manner will also satisfy (ar. N) which is not required in Theorem 7.

We shall prove Theorem 8 by induction on $n \geqq 1$.

PROOF OF THEOREM 8. *The case* $n = 1$. In this case $|b| + n \geqq 2$ implies that $b \neq \emptyset$. We also have $T = \{\{i\} : i \in T^*\}$ and $|T^*| \geqq 2$. To save notation, let us assume that 1 and 2 are in T^*. We also set $b1 = b \cup \{1\}$ and $b2 = b \cup \{2\}$. So, by our assumptions, φ_{bi} are interior for $i = 1, 2$.

(4.1) For $U \subset\subset X$ we use the symbol $V \subset_{(bi)} U$ ($i = 1, 2$) to state that $V \subset\subset U$, and that there exists a cube $D = \Pi_{r \in bi} D_r \subset\subset \Pi_{r \in bi} Y_r$, with $D_r \subset\subset Y_r$, so that $\varphi_{bi}(V) \subset D \subset \varphi_{bi}(U)$.

We claim that

(4.2) If $V \subset_{(bi)} U$ and $\beta \in \varphi_b(V)$ and $\alpha \in \varphi_i(V)$ then there exists some $x \in U$ with $\varphi_b(x) = \beta$ and $\varphi_i(x) = \alpha$.

Indeed, since $\varphi_{bi}(V) \subset \Pi_{r \in bi} D_r = D$, $(\beta, \alpha) \in (\Pi_{r \in b} D_r) \times D_i = D$ and since $D \subset \varphi_{bi}(U)$, there must be some $x \in U$ so that $\varphi_{bi}(x) = (\beta, \alpha)$.

We also have

(4.3) For every $U \subset\subset X$ there exists some V such that $V \subset_{(bi)} U$, $i = 1, 2$.

Indeed, since φ_{bi} is interior and U is open, the interior of $\varphi_{bi}(U)$ in $\Pi_{r \in bi} Y_r$ is nonempty. So, by the definition of the product topology, there must be some cube $D = \Pi_{r \in bi} D_r$, with $D_r \subset\subset Y_r$, such that $D \subset \varphi_{bi}(U)$, and we may take $V = U \cap \varphi_{bi}^{-1}(D)$.

Let us see now that the case $n = 1$ of Theorem 8 holds with the constant $c = c(1, |T^*|) = 2$ (i.e., in this case c does not depend on $|T^*|$). So let $L \geqq 1$ and $U \subset\subset X$ be given. We construct inductively a sequence $\{U_l\}_{l=1}^{L}$ of open subsets of X so that $U = U_L$ and also

(4.4) $$U_{2r-1} \subset_{(b1)} U_{2r}, \quad U_{2r} \subset_{(b2)} U_{2r+1}, \qquad r = 1, 2, \ldots .$$

This is done as follows: set $U_L = U$. Apply (4.3) to obtain some $U_{L-1} \subset_{(b1)} U_L$. Another application of (4.3) implies the existence of a $U_{L-2} \subset_{(b2)} U_{L-1}$, and still another application of (4.3) provides us with a $U_{L-3} \subset_{(b1)} U_{l-2}$, and we continue by an obvious induction. Note that if L is even, then (4.4) holds. If L is odd we must begin the process with $b2$ instead of $b1$ (i.e., $U_{L-1} \subset_{(b2)} U_L$) in order to obtain (4.4). Set $V = U_1$, and we claim that V satisfies Theorem 8 (w.r.t. U and L). We have

$$V = U_1 \subset_{(b1)} U_2 \subset_{(b2)} U_3 \subset_{(b1)} U_4 \subset_{(b2)} U_5 \subset \cdots .$$

Let $\beta^+ = \{\beta_l^+\}_{l=1}^{L^+}$ and $\beta^- = \{\beta_l^-\}_{l=1}^{L^-}$ be disjoint sequences in $\varphi_b(V)$, with $L^+ + L^- = L$ and $|L^+ - L^-| \leqq 1$. So $\beta_1^+ \in \varphi_b(V) = \varphi_b(U_1)$. Hence there exists some $x_1 \in U_1$ with $\varphi_b(x_1) = \beta_1^+$. Clearly, $\varphi_1(x_1) \in \varphi_1(U_1)$ and $\beta_1^- \in \varphi_b(U_1)$ hence, as $U_1 \subset_{(b1)} U_2$, it follows from (4.2) that there exists a point $x_2 \in U_2$ with $\varphi_1(x_1) = \varphi_1(x_2)$ and $\varphi_b(x_2) = \beta_1^-$. Now, $\varphi_2(x_2) \in \varphi_2(U_2)$ and $\beta_2^+ \in \varphi_b(U_2)$; hence, since $U_2 \subset_{(b2)} U_3$, there exists a point $x_3 \in U_3$ with $\varphi_2(x_2) = \varphi_2(x_3)$, and $\varphi_b(x_3) = \beta_2^+$. We continue inductively, and construct points $x_j \in U_j$, $1 \leqq j \leqq L$ so that

$$\tag{4.5} \begin{gathered} \varphi_1(x_{2r-1}) = \varphi_1(x_{2r}), \quad \varphi_2(x_{2r}) = \varphi_2(x_{2r+1}), \\ \beta_r^+ = \varphi_b(x_{2r-1}), \quad \beta_r^- = \varphi_b(x_{2r}), \qquad r = 1, 2, \dots . \end{gathered}$$

(Note that in (4.4) and (4.5) we did not mention the upper bound for r, since it depends on the parity of L.)

Set $\varepsilon(j) = (-1)^{j+1}$, and let $\mu = \sum_{j=1}^{L} \varepsilon(j)\delta_{x_j}$. We claim that μ is a normal array which satisfies Theorem 8.

Note first that $\varphi_b(\{x_j\}_{\varepsilon(j)=\pm 1}) = \beta^{\pm}$. Indeed, $\{x_j\}_{\varepsilon(j)=1} = \{x_{2r-1}\}_{r\geqq 1}$ and, by (4.5), $\varphi_b(x_{2r-1}) = \beta_r^+$; also $\{x_j\}_{\varepsilon(j)=-1} = \{x_{2r}\}_{r\geqq 1}$ and $\varphi_b(x_{2r}) = \beta_r^-$. (ar.1) is satisfied trivially. $\beta^+ \cap \beta^- = \varnothing$ implies that $\{x_{2r-1}\}_{r\geqq 1} \cap \{x_{2r}\}_{r\geqq 1} = \varnothing$, and thus $\|\mu\| = |\beta^+| + |\beta^-| = L^+ + L^- = L$ and (ar.2) follows.

To demonstrate (ar.3) we must identify the subsets L_i of $\{1, 2, \dots, L\}$, $i \in T^*$. So let $L_1 = \bigcup_{r\geqq 1}\{2r-1, 2r\}$, $L_2 = \bigcup_{r\geqq 1}\{2r, 2r+1\}$, and $L_i = \varnothing$ for $i \in T^* \setminus \{1,2\}$. (Note that the union in the definition of L_1 and L_2 is taken over those values of r for which the corresponding pairs are contained in $\{1, 2, \dots, L\}$. Thus, e.g., if L is even then $L_1 = \bigcup_{r=1}^{L/2}\{2r-1, 2r\}$ and $L_2 = \bigcup_{r=1}^{L/2-1}\{2r, 2r+1\}$.) In this setting it is convenient to check (ar. 3.1′): L_i $(i = 1, 2)$ is actually presented in terms of its decomposition E_i, $E_1 = \{\{2r-1, 2r\}\}$ and $E_2 = \{\{2r, 2r+1\}\}$. Since $\varepsilon(j) = (-1)^{j+1}$ we have that for $\{j, j'\} \in E_i$, $\varepsilon(j)\varepsilon(j') = -1$ and by (4.5) also $\varphi_i(x_j) = \varphi_i(x_{j'})$, $i = 1, 2$, and (ar.3.1′) follows. To check (ar.3.2), note first that, for $i \not\in \{1, 2\}$, $L_i = \varnothing$, and thus for all $j \in \{1, 2, \dots, L\}$, $\sigma(j) = \{i : i \in T^*, j \in L_i\}$ satisfies $|\sigma(j)| \leqq 2$. Also, for $2 \leqq j \leqq L-1$, $\sigma(j) = \{1, 2\}$, i.e.,

$$|\{j : |\sigma(j)| = 2n = 2\}| \geqq L - 2 = \|\mu\| - c$$

(recall that $c = 2$) and (ar.3.2) follows. We still have to check (ar. N). Let σ be a subset of T^*. If $\sigma \neq \{1, 2\}$ then $\{j : 1 \leqq j \leqq L,\ \sigma(j) = \sigma\} \subset \{1, L\}$, and thus $|\sum_{j:\sigma(j)=\sigma} \varepsilon(j)| \leqq 2 = c$. For $\sigma = \{1, 2\}$, $\{j : \sigma(j) = \sigma\} = \{2, 3, \dots, L-1\}$, hence

$$\left| \sum_{j:\sigma(j)=\sigma} \varepsilon(j) \right| = \left| \sum_{j=2}^{L-1} (-1)^{j+1} \right| \leqq 1 < c$$

which settles (ar. N), and concludes the proof of Theorem 8 for $n = 1$.

We proceed towards the inductive step. The following concept and lemma will be applied there

DEFINITION 4.2. Let X be a set, let $\{\varphi_i\}_{i=1}^k$ be functions on X, let T^* and $b \neq \emptyset$ be two disjoint subsets of $\{1,2,\ldots,k\}$, let $n \geqq 1$ be an integer, and let $c > 0$. A pair $\mu = \Sigma_{j=1}^L \varepsilon(j)\delta_{x_j}$ and $\tilde{\mu} = \Sigma_{j=1}^L \tilde{\varepsilon}(j)\delta_{\tilde{x}_j}$ of *normal arrays* w.r.t. $F = \{\varphi_i\}_{i \in T^*}$, of order n, and constant c, is called a *double array* w.r.t. b if $\{x_j\}_{j=1}^L \cap \{\tilde{x}_j\}_{j=1}^L = \emptyset$ and

$$(4.6) \qquad \varphi_b(\{x_j\}_{\varepsilon(j)=1}) = \varphi_b(\{\tilde{x}_j\}_{\tilde{\varepsilon}(j)=-1}), \quad \varphi_b(\{x_j\}_{\varepsilon(j)=-1}) = \varphi_b(\{\tilde{x}_j\}_{\tilde{\varepsilon}(j)=1}).$$

LEMMA 4.1. *Let X, Y_i, $1 \leqq i \leqq k$, $\{\varphi_i\}_{i=1}^k$, b, T and $n \geqq 1$ be as in Theorem 8. Assume also that $b \neq \emptyset$, and that Theorem 8 holds for this n. Then every $U \subset\subset X$ contains a double array μ, $\tilde{\mu}$, of order n, with the constant c guaranteed by Theorem 8, w.r.t. $F = \{\varphi_i\}_{i \in T^*}$ and b, of arbitrary norm L.*

PROOF. Let $U \subset\subset X$ be given. Fix some $a \in T$. Let ab denote $a \cup b$. Then φ_{ab} is interior. Thus $\varphi_{ab}(U)$ has a nonempty interior in $Y_a \times Y_b$ (where $Y_a = \Pi_{i \in a} Y_i$ and $Y_b = \Pi_{i \in b} Y_i$). Hence there exists some $D = A \times B \subset \varphi_{ab}(U)$, where D is an $n + |b|$-cube in Y_{ab}, with A an n-cube in Y_a and B a $|b|$-cube in Y_b. (Note that by an "m-cube" we mean a set of the form $\Pi_{i \in d} D_i \subset Y_d$, where $|d| = m$ and $D_i \subset\subset Y_i$.)

Let A' and A'' be two disjoint n-cubes in A. (A' and A'' exist by our assumption of the spaces Y_i.) Set $D' = A' \times B$ and $U' = U \cap \varphi_{ab}^{-1}(D')$.

Let $L \geqq 1$ be any integer. By our assumption there exists some $V' \subset\subset U'$ which satisfies the conclusion of Theorem 8 for this given L. Let $B'' \subset \varphi_b(V')$ be a $|b|$-cube (B'' exists since φ_b is clearly interior.) Set $D'' = A'' \times B'' \subset A'' \times B \subset \varphi_{ab}(U)$, and also $U'' = U \cap \varphi_{ab}^{-1}(D'')$. Apply Theorem 8 once again to find some $V'' \subset U''$ which satisfies its conclusion. We have

$$(4.7) \qquad U', U''' \subset U, \quad U' \cap U'' = \emptyset, \quad \varphi_b(V''') \subset \varphi_b(V').$$

Let now β_1 and β_2 be two disjoint subsets of $\varphi_b(V''')$ with $|\beta_1| + |\beta_2| = L$ and $||\beta_1| - |\beta_2|| \leqq 1$. By the choice of V'', there exists a normal array $\mu = \Sigma_{j=1}^L \varepsilon(j)\delta_{x_j}$ of order n, constant c and norm L w.r.t. $F = \{\varphi_i\}_{i \in T^*}$ in U'', so that $\varphi_b(\{x_j\}_{\varepsilon(j)=1}) = \beta_1$ and $\varphi_b(\{x_j\}_{\varepsilon(j)=-1}) = \beta_2$. By (4.7), β_1 and β_2 are also subsets of $\varphi_b(V')$ and hence we can find a normal array $\tilde{\mu} = \Sigma_{j=1}^L \tilde{\varepsilon}(j)\delta_{\tilde{x}_j}$ in U', so that $\varphi_b(\{\tilde{x}_j\}_{\tilde{\varepsilon}(j)=1}) = \beta_2$ while $\varphi_b(\{\tilde{x}_j\}_{\tilde{\varepsilon}(j)=-1}\} = \beta_1$. Thus we have

$$\varphi_b(\{x_j\}_{\varepsilon(j)=1}) = \beta_1 = \varphi_b(\{\tilde{x}_j\}_{\tilde{\varepsilon}(j)=-1}\} \quad \text{and} \quad \varphi_b(\{x_j\}_{\varepsilon(j)=-1}) = \beta_2 = \varphi_b(\{\tilde{x}_j\}_{\tilde{\varepsilon}(j)=1}\},$$

i.e., (4.6) is satisfied, and the lemma follows.

PROOF OF THEOREM 8 — *The Inductive Step.* Let $n \geqq 2$ be an integer, and assume that Theorem 8 holds for all lesser values than n. Let X, Y_i, $\{\varphi_i\}_{i=1}^k$, T and b be as in Theorem 8. For $i \in T^*$ set $bi = b \cup \{i\}$. Let T_i, $i \in T^*$ be the tree of order $n-1$, and type $\{2,4,\dots,2(n-1)\}$, which corresponds to i by Definition 2.1 of a tree. Then $bi \cap T_i^* = \emptyset$ and the induction hypothesis can be applied to the tree T_i, with bi replacing b. (Note that for $a \in T_i$ ($|a| = n-1$), $bi \cup a = b \cup \{\{i\} \cup a\}$, where $\{i\} \cup a \in T$ by Definition 2.1, and thus $\varphi_{bi\cup a}$ is interior by the assumptions of Theorem 8.) Hence, for each $i \in T^*$ there exists a constant $c(n-1, |T_i^*|)$ so that Theorem 8 holds with this constant, w.r.t. bi and T_i. Clearly, $|T^*| > |T_i^*|$ for all $i \in T^*$, thus

$$c = c(n-1, |T^*|) \geqq c(n-1, |T_i^*|) \qquad \text{for all } i \in T^*,$$

and it follows that Theorem 8 holds with this value of c for all bi and T_i, $i \in T^*$. We shall prove that Theorem 8 holds for b and T with the constant

$$c(n, |T^*|) = 9|T^*|^2 2^{|T^*|} c(n-1, |T^*|).$$

Given $i \in T^*$, $L \geqq 1$ and $U \subset\subset X$, the induction hypothesis guarantees the existence of some $V \subset\subset U$ which satisfies Theorem 8, w.r.t. bi, T_i, the constant $c = c(n-1, |T^*|)$, and (the norm) L. We use the symbol $V <_{(bi,T_i,L)} U$ to denote that V satisfies the above.

Let now L be a positive integer and $U \subset\subset X$. We shall construct a subset $V \subset\subset U$ which satisfies Theorem 8, with the constant $c(n, |T^*|)$ mentioned above.

Let m be the largest even integer so that $m^n \leqq L$. One easily checks that

$$L - m^n < 2^{2n} m^{n-1} \leqq 2^{|T^*|} L^{(n-1)/n}. \tag{4.8}$$

(4.9) CLAIM. *For every $U \subset\subset X$, there exist open subsets $U_1, U_2, \dots, U_{|T^*|}$ and S of U and open sets $V_1, V_2, \dots, V_{|T^*|}$ and W so that:*

(i) *$U_1, U_2, \dots, U_{|T^*|}$ and S are mutually disjoint.*

(ii) $V_i <_{(bi,T_i,m^{n-1})} U_i$, $i \in T^*$.

(iii) $\varphi_{bi}(S) \subset \varphi_{bi}(V_i)$, $i \in T^*$.

(iv) $W \subset_{(bi)} S$, $i \in T^*$.

(Recall that by (4.1), $W \subset_{(bi)} S$ means that $W \subset\subset S$ and that there exist some $|b|+1$ cube D in $Y_{bi} = \Pi_{r\in bi} Y_r$ so that $\varphi_{bi}(W) \subset D \subset \varphi_{bi}(S)$, $D = D_b \times D_i$, $D_b \subset\subset Y_b$, $D_i \subset\subset Y_i$. Note also that in this stage of the proof b may be empty. If $b = \emptyset$ then (iv) is meaningless, and we may take $W = S$. Clearly, $bi \neq \emptyset$ for all $i \in T^*$.)

PROOF OF (4.9). Let $U \subset\subset X$ be given. To save notation let us assume that $T^* = \{i\}_{i=1}^{|T^*|}$. Pick some $a_1 \in T_1$. Then $\varphi_{a_1 b1}$ is interior (where, of course, $a_1 b1 = a_1 \cup b1$). Let $A_1 \times B_1 \subset \varphi_{a_1 b1}(U)$ be a $(|b| + n)$-cube, with $A_1 \subset\subset Y_{a_1}$ an $(n-1)$-cube and $B_1 \subset\subset Y_{b1}$ a $(|b|+1)$-cube. (Recall that $Y_d = \Pi_{i \in d} Y_i$.) Let A_1' and A_1'' be two disjoint $(n-1)$-cubes in A_1. (A_1' and A_1'' exist by our assumption on the spaces Y_i.)

Set

$$U_1 = \varphi_{a_1 b1}^{-1}(A_1' \times B_1) \cap U.$$

Apply the induction hypothesis to find some $V_1 \subset\subset U_1$ so that $V_1 <_{(b1, T_1, m^{n-1})} U_1$. Thus, in particular, $\varphi_{b1}(V_1) \subset \varphi_{b1}(U_1) = B_1$. Let $Z \subset\subset \varphi_{b1}(V_1)$. (Such a Z exists since V_1 is open and φ_{b1} is interior.) Set

$$S_1 = \varphi_{a_1 b1}^{-1}(A_1'' \times Z) \cap U.$$

Since $A_1' \cap A_1'' = \varnothing$, we also have $S_1 \cap U_1 = \varnothing$, and also $\varphi_{b1}(S_1) \subset \varphi_{b1}(V_1)$. Altogether we have:

$$\text{(4.10)} \qquad \begin{gathered} U_1 \text{ and } S_1 \text{ are disjoint open subsets of } U_1, \\ V_1 <_{(b1, T_1, m^{n-1})} U_1 \quad \text{and} \quad \varphi_{b1}(S_1) \subset \varphi_{b1}(V_1). \end{gathered}$$

Pick now some $a_2 \in T_2$, and operate on S_1 with $a_2 b2$ as we have operated on U with $a_1 b1$. By doing this we obtain

$$\text{(4.11)} \qquad \begin{gathered} U_2 \text{ and } S_2 \text{ are disjoint open subsets of } S_1, \\ V_2 <_{(b2, T_2, m^{n-1})} U_2 \quad \text{and} \quad \varphi_{b2}(S_2) \subset \varphi_{b2}(V_2). \end{gathered}$$

Clearly, U_1, U_2 and S_2 are mutually disjoint, and since $S_2 \subset S_1$ we also have

$$\text{(4.12)} \qquad \varphi_{b1}(S_2) \subset \varphi_{b1}(S_1) \subset \varphi_{b1}(V_1).$$

Now pick some $a_3 \in T_3$, and operate on S_2 as above with $(a_3 b3)$, to obtain $U_3, S_3 \subset S_2$ and $V_3 <_{(b3, T_3, m^{n-1})} U_3$, and continue by an obvious induction with $i = 4, 5, \ldots, |T^*|$. At the $|T^*|$ step we obtain $U_{|T^*|}$, $S_{|T^*|} \subset S_{|T^*|-1}$, and

$$V_{|T^*|} <_{(b|T^*|, T_{|T^*|}, m^{n-1})} U_{|T^*|},$$

and we set $S_{|T^*|} = S$. If $b = \varnothing$ we set $W = S$. If $b \neq \varnothing$, apply the fact that φ_{bi} is interior for all $i \in T^*$, to construct a sequence $W_{|T^*|}, W_{|T^*|-1}, \ldots, W_2, W_1$ so that

$$W_1 \subset_{(b1)} W_2 \subset_{(b2)} W_3 \subset_{(b3)} W_4 \subset \cdots \subset W_{|T^*|} \subset_{(b|T^*|)} S$$

(see (4.3)). Set $W = W_1$. Then clearly $W_1 \subset_{(bi)} S$ for all $i \in T^*$, i.e., (iv) is satisfied. (iii) is satisfied too as shown in (4.10) and (4.11); and so is (ii). (i) holds, since in each step of the construction $U_1, U_2, \ldots, U_i, S_i$ are disjoint, and U_{i+1}, S_{i+1} are disjoint subsets of S_i. This proves (4.9).

We return now to our given $U \subset\subset X$. Applying (4.9) m times, we construct in U open sets U_i^j, V_i^j, S^j and W^j, $1 \leqq j \leqq m$, $1 \leqq i \leqq |T^*|$, by induction on $j = m, m-1, \ldots, 2, 1$, as follows: First, for $j = m$, apply (4.9) on U, to obtain U_i^m, V_i^m, $1 \leqq i \leqq |T^*|$, S^m and W^m in U, which satisfy (i), (ii), (iii) and (iv) of (4.9). Assume that U_i^{j+1}, V_i^{j+1}, S^{j+1} and W^{j+1} have been constructed. By applying (4.9) one more time on W^{j+1}, we obtain, in W^{j+1}, U_i^j, V_i^j, $i \in T^*$, S^j and W^j. From this construction it then follows that

(4.13.0) W^j, S^j, V_i^j and U_i^j are open subsets of W^{j+1}, $\quad 1 \leqq j \leqq m-1, \quad i \in T^*$.

(4.13.1) The sets U_i^j, $\quad 1 \leqq j \leqq m, \quad i \in T^*$ are mutually disjoint.

(4.13.2) $$V_i^j <_{(bi, T_i, m^{n-1})} U_i^j, \qquad 1 \leqq j \leqq m, \quad i \in T^*.$$

(4.13.3) $$W^j \subset_{(bi)} S^j, \qquad 1 \leqq j \leqq m, \quad i \in T^*.$$

(4.13.4) $$\varphi_{bi}(S^j) \subset \varphi_{bi}(V_i^j), \qquad 1 \leqq j \leqq m, \quad i \in T^*.$$

Set $V = W^1$, and we claim that V satisfies Theorem 8 (i.e., $V <_{(b,T,L)} U$).

To see this, let $\beta^+ = \{\beta_l^+\}_{l=1}^{L^+}$ and $\beta^- = \{\beta_l^-\}_{l=1}^{L^-}$ be two disjoint sequences in $\varphi_b(V)$, with $L^+ + L^- = L$ and $|L^+ - L^-| \leqq 1$. We have to show the existence of a normal array $\mu = \sum_{l=1}^{L} \varepsilon(l)\delta_{x_l}$ in U, of order n, constant $c(n, |T^*|) = 9|T^*|^2 2^{|T^*|}c$, and norm L, w.r.t. $F = \{\varphi_i\}_{i \in T^*}$, such that $\varphi_b(\{x_l\}, \varepsilon(l) = \pm 1) = \beta^{\pm}$.

Before presenting the details of the proof, which is lengthy and complicated, we wish to comment on its general strategy. The sequence $\{x_l\}$, $1 \leqq l \leqq L$ will consist of $m+1$ subsequences M_j, $1 \leqq j \leqq m+1$. The first m M_j's will be constructed by induction on $j = 1, 2, \ldots, m$ so that the length of M_j is m^{n-1} and, rougly speaking, each M_j is decomposed into subsequences, most of which are normal arrays of order $n-1$. Together with the points of M_j, for each $x_l \in M_j$, we shall also construct the "sign function" $\varepsilon(l) = \pm 1$, and in the inductive procedure we shall see to it that for "many" x_l's in M_j, there will correspond some $x_{l'} \in M_{j+1}$, with $\varepsilon(l)\varepsilon(l') = -1$ and $\varphi_i(x_l) = \varphi_i(x_{l'})$ for some $i \in T^*$, which, after "filling up" the amount by constructing M_{m+1}, will imply that $\mu = \sum_{l=1}^{L} \varepsilon(l)\delta_{x_l}$ is a normal array as we wish. As mentioned, the sequences M_j, $j = 1, 2, \ldots, m$ will be constructed inductively, and such that $M_j \subset \bigcup_{i \in T^*} U_i^j$ (which by (4.13.1) guarantees that the M_j's are mutually disjoint). It turns out that the structure of M_1 does not reveal the whole complexity of the structure of M_j for $j \geqq 2$; and thus the presentation of the inductive step (in the construction of the M_j's) right after the construction of M_1, though possible, might seem unnatural to the reader. Hence we have decided to present the construction of M_1 first, then to show how M_2 is derived from M_1, and then to describe the inductive derivation of M_j from M_{j-1}. Clearly, some of the features in $M_{j-1} \Rightarrow M_j$ appear also in $M_1 \Rightarrow M_2$, and thus will be presented twice. Still, we feel that this approach will make the proof more accessible to the reader.

To construct M_1, we pick some $i \in T^*$, e.g., $i = 1$. We also set

$$B_1^+ = \{\beta_l^+\}_{l=1}^{m^{n-1}/2} \quad \text{and} \quad B_1^- = \{\beta_l^-\}_{l=1}^{m^{n-1}/2}.$$

(Recall that m is an even integer.) Let

$$\alpha^+ = \{\alpha_l^+\}_{l=1}^{m^{n-1}/2} \quad \text{and} \quad \alpha^- = \{\alpha_l^-\}_{l=1}^{m^{n-1}/2}$$

be two disjoint sequences in $\varphi_1(V)$. Since $V = W^1 \subset_{(b1)} S^1$ (by (4.13.3)) the sequences

$$\delta_1^+ = \{(\beta_l^+, \alpha_l^+)\}_{l=1}^{m^{n-1}/2} \quad \text{and} \quad \delta_1^- = \{(\beta_l^-, \alpha_l^-)\}_{l=1}^{m^{n-1}/2}$$

are both in $\varphi_{b1}(S^1)$. (Note that $\delta_1^+ \cap \delta_1^- = \varnothing$. Our proof covers also the case $b = \varnothing$, and in that case (where there are no β's) $\delta_1^\pm = \alpha^\pm$, and the α's were selected to be disjoint sequences.) By (4.13.4) $\varphi_{b1}(S^1) \subset \varphi_{b1}(V_1^1)$, i.e., $\delta_1^\pm \subset \varphi_{b1}(V_1^1)$. Since by (4.13.2) $V_1^1 <_{(b1, T_1, m^{n-1})} U_1^1$, we can find a normal array $\nu_1 = \sum_{l=1}^{m^{n-1}} \varepsilon(l)\delta_{x_l}$ in U_1^1, of order $n-1$, and constant c, w.r.t. $\{\varphi_i\}_{i \in T_1^*}$, such that $\varphi_{b1}(\{x_l\}_{\varepsilon(l)=\pm 1}) = \delta_1^\pm$. We take $M_1 = \{x_l\}_{l=1}^{m^{n-1}}$, and the signs $\varepsilon(l)$ for $1 \leqq l \leqq m^{n-1}$ which correspond to the array ν_1 will be taken as the signs in μ too. In this way we construct $M_1 = \{x_l\}_{l=1}^{m^{n-1}}$, and $\varepsilon(l)$, $1 \leqq l \leqq m^{n-1}$. Note that $\varphi_b(\{x_l\}_{\varepsilon(l)=\pm 1}) = B_1^\pm$.

To construct M_2, we shall first have to reorder M_1. Actually, we shall reorder the indices $1 \leqq l \leqq m^{n-1}$. Let $\tilde{\tau} : \{\sigma \subset T_1^*, |\sigma| \leqq 2n-1\} \to T^*$ be a function such that $\tilde{\tau}(\sigma) \notin \sigma$. (Such a function exists, since $|T^*| \geqq 2n$, as T is a tree of type $(2, 4, \ldots, 2n)$.) $\nu_1 = \sum_{l=1}^{m^{n-1}} \varepsilon(l)\delta_{x_l}$ is an array w.r.t. $\{\varphi_i\}_{i \in T_1^*}$ and, as such, to each $1 \leqq l \leqq m^{n-1}$, there corresponds a subset $\sigma(l) \subset T_1^*$ with $|\sigma(l)| \leqq 2(n-1)$, by (ar.3.2). Note that $1 \notin \sigma(l)$, since $T_1^* \subset T^* \setminus \{1\}$. Set $\tau(l) = \tilde{\tau}(\{1\} \cup \sigma(l))$, $1 \leqq l \leqq m^{n-1}$. For $i \in T^*$ let

$$N_i = \tau^{-1}(i) = \{l : 1 \leqq l \leqq m^{n-1}, \tau(l) = i\}.$$

The N_i's are disjoint sets of indices. (Note that $N_1 = \varnothing$ since $\tilde{\tau}(\sigma) \notin \sigma$.)

(4.14) CLAIM. $|\sum_{l \in N_i} \varepsilon(l)| \leqq 2^{|T^*|} c m^{n-2}$, *for all* $i \in T^*$.

Indeed, let $i \in T^*$. Then

$$N_i = \{l : \tau(l) = \tilde{\tau}(\{1\} \cup \sigma(l)) = i\} = \bigcup_{\sigma} \{l : \sigma(l) = \sigma\}$$

where the union is taken over those subsets σ of T_1^* such that $\tilde{\tau}(\{1\}\cup\sigma)=i$. Clearly, the number of such sets σ is less than $2^{|T_1^*|}<2^{|T^*|}$. Hence

$$\left|\sum_{l\in N_i}\varepsilon(l)\right|=\left|\sum_{\sigma:\tilde{\tau}(\{1\}\cup\sigma)=i}\ \sum_{l:\sigma(l)=\sigma}\varepsilon(l)\right|\leqq\sum_{\sigma:\tilde{\tau}(\{1\}\cup\sigma)=i}\left|\sum_{l:\sigma(l)=\sigma}\varepsilon(l)\right|\leqq 2^{|T^*|}\cdot cm^{n-2},$$

since $|\Sigma_{l:\sigma(l)=\sigma}\,\varepsilon(l)|\leqq cm^{n-2}$ by (ar. N) as ν_1 is a *normal* array of order $n-1$ and norm m^{n-1}.

Now, we decompose each N_i into 3 sets ${}^1N_i$, ${}^2N_i$ and ${}^3N_i$ as follows:

Selection of ${}^1N_i$: If $b\neq\emptyset$ we take ${}^1N_i=\emptyset$. If $b=\emptyset$, we select a maximal number of disjoint pairs $\{l,l'\}\subset N_i$, wjth $\varepsilon(l)\cdot\varepsilon(l')=-1$ and $\varphi_i(x_l)=\varphi_i(x_{l'})$, and let ${}^1N_i$ be the union of those pairs. Note that (if $b=\emptyset$) then, for l, l' in $N_i\setminus{}^1N_i$, $\varepsilon(l)\cdot\varepsilon(l')=-1$ implies that $\varphi_i(x_l)\neq\varphi_i(x_{l'})$. Clearly, $\Sigma_{l\in{}^1N_i}\,\varepsilon(l)=0$.

Selection of ${}^2N_i$ *and* ${}^3N_i$: ${}^2N_i$ is selected to be a subset of $N_i\setminus{}^1N_i$ with maximal cardinality, so that $\Sigma_{l\in{}^2N_i}\,\varepsilon(l)=0$. (For example, if

$$|\{l\in N_i\setminus{}^1N_i:\varepsilon(l)=1\}|\leqq|\{l\in N_i\setminus{}^1N_i:\varepsilon(l)=-1\}|,$$

then we take ${}^2N_i=\{l\in N_i\setminus{}^1N_i:\varepsilon(l)=1\}\cup P$, where $P\subset\{l\in N_i\setminus{}^1N_i:\varepsilon(l)=-1\}$ is a subset with $|P|=|\{l\in N_i\setminus{}^1N_i,\varepsilon(l)=1\}|$.)

We also set ${}^3N_i=N_i\setminus({}^1N_i\cup{}^2N_i)$. The following then holds:

(4.15) $$\sum_{l\in{}^2N_i}\varepsilon(l)=0\quad\text{and also}\quad\sum_{l\in{}^1N_2}\varepsilon(l)=0.$$

(4.16) $$\left|\sum_{l\in{}^3N_i}\varepsilon(l)\right|=|{}^3N_i|\quad\text{(i.e., the elements of }{}^3N_i\text{ have constant signs).}$$

Indeed, if l and l' are in ${}^3N_i$ and $\varepsilon(l)\cdot\varepsilon(l')=-1$, then we can add l and l' to ${}^2N_i$ without harming (4.15), and since ${}^2N_i$ has been selected to be a maximal set with (4.15), (4.16) follows.

(4.17) $$|{}^3N_i|\leqq 2^{|T^*|}cm^{n-2}.$$

Indeed, by (4.14)

$$2^{|T^*|}cm^{n-2}\geqq\left|\sum_{l\in N_i}\varepsilon(l)\right|=\left|\sum_{l\in{}^1N_i}\varepsilon(l)+\sum_{l\in{}^2N_i}\varepsilon(l)+\sum_{l\in{}^3N_i}\varepsilon(l)\right|$$
$$=\left|\sum_{l\in{}^3N_i}\varepsilon(l)\right|\qquad\text{by (4.15).}$$

We also have

(4.18) $$\sum_{i\in T^*}\sum_{l\in{}^3N_i}\varepsilon(l)=0.$$

This follows from the fact that

$$\sum_{i\in T^*}\sum_{l\in {}^3N_i}\varepsilon(l)=\sum_{i\in T^*}\sum_{l\in N_i}\varepsilon(l)-\Big(\sum_{i\in T^*}\sum_{l\in {}^1N_i}\varepsilon(l)+\sum_{i\in T^*}\sum_{l\in {}^2N_i}\varepsilon(l)\Big).$$

Each of the sums $\Sigma_{l\in {}^1N_i}\varepsilon(l)$, $\Sigma_{l\in {}^2N_i}\varepsilon(l)$ is 0 by (4.15), while $\Sigma_{i\in T^*}\Sigma_{l\in N_i}\varepsilon(l)=\Sigma_{l\in M_1}\varepsilon(l)=0$, since

$$|\{l:1\leqq l\leqq m^{n-1},\varepsilon(l)=1\}|=\tfrac12 m^{n-1}=|\{l:1\leqq l\leqq m^{n-1},\varepsilon(l)=-1\}|,$$

which follows from the fact that the x_l's which correspond to the l's with $\varepsilon(l)=\pm1$ are mapped by φ_{bi} onto $\delta^{\pm}$, and both δ^+ and δ^- are sequences of length $\frac12 m^{n-1}$. Finally

If $b=\varnothing$, then

(4.19) $$\varphi_i(\{x_l\},l\in N_i\setminus{}^1N_i,\varepsilon(l)=1)\cap\varphi_i(\{x_l\},l\in N_i\setminus{}^1N_i,\varepsilon(l)=-1)=\varnothing,$$

which follows from the selection of ${}^1N_i$. (In particular (4.19) holds for ${}^2N_i\subset N_i\setminus{}^1N_i$.)

We come now to the construction of M_2. For each $i\in T^*$ we shall construct a sequence $M_{2(i)}$ in U_i^2, and take $M_2=\bigcup_{i\in T^*}M_{2(i)}$ (in some ordering.) Each $M_{2(i)}$ will be constructed as a union $M_{2(i)}={}^1M_{2(i)}\cup{}^2M_{2(i)}\cup{}^3M_{2(i)}$. So, fix some $i\in T^*$. We shall first construct ${}^2M_{2(i)}$. For a subset P of $\{1,2,\dots,m^{n-1}\}$ let

$$P^+=\{l:l\in P,\varepsilon(l)=1\}\quad\text{and}\quad P^-=\{l:l\in P,\varepsilon(l)=-1\}.$$

By (4.5), $|{}^2N_i^+|=|{}^2N_i^-|=\frac12|{}^2N_i|$. Let ${}^2B^+_{2(i)}=\{\beta_r^+\}_{r=1}^{|{}^2N_i|/2}$ be a subsequence of $\beta^+\setminus B_1^+$ and ${}^2B^-_{2(i)}=\{\beta_r^-\}_{r=1}^{|{}^2N_i|/2}$ a subsequence of $\beta^-\setminus B_1^-$. Then ${}^2B^{\pm}_{2(i)}\subset\varphi_b(V)=\varphi_b(W^1)\subset\varphi_b(W^2)$ by (4.13.0) and since $M_1\subset U_1^i\subset W^2$, we also have that $\varphi_i(\{x_l\}_{l\in {}^2N_i^+})\subset\varphi_i(W^2)$ and $\varphi_i(\{x_l\}_{l\in {}^2N_i^-})\subset\varphi_i(W^2)$. By (4.13.2), $W^2\subset_{(bi)}S^2$. Hence, we can select in $\varphi_{bi}(S^2)$ two sequences ${}^2\delta^+_{2(i)}$ and ${}^2\delta^-_{2(i)}$ of length $\frac12|{}^2N_i|$ each, such that

$${}^2\delta^+_{2(i)}=\{(\beta_r^+,\alpha_r^-)\}_{r=1}^{|{}^2N_i|/2}\quad\text{and}\quad {}^2\delta^-_{2(i)}=\{(\beta_r^-,\alpha_r^+)\}_{r=1}^{|{}^2N_i|/2}$$

where

$$\{\alpha_r^-\}_{r=1}^{|{}^2N_i|/2}=\varphi_i(\{x_l\}_{l\in {}^2N_i^-})\quad\text{and}\quad\{\alpha_r^+\}_{r=1}^{|{}^2N_i|/2}=\varphi_i(\{x_l\}_{l\in {}^2N_i^+}\}.$$

(In other words, ${}^2\delta^+_{2(i)}$ is a sequence in $\varphi_{bi}(S^2)\subset Y_b\times Y_i$, such that if we project it into Y_b we get the sequence ${}^2B^+_{2(i)}$, while if we project it into Y_i we obtain the sequence $\varphi_i(\{x_l\}_{l\in {}^2N_i^-})$; ${}^2\delta^-_{2(i)}$ is projected to ${}^2B^-_{2(i)}$ in Y_b and to $\varphi_i(\{x_l\}_{l\in {}^2N_i^+})$ in Y_i. As mentioned above, the existence of the ${}^2\delta^{\pm}_{2(i)}$ follows from (4.13.3) and (4.2).)

Note that ${}^2\delta^+_{2(i)}\cap{}^2\delta^-_{2(i)}=\varnothing$. This follows from (4.19) if $b=\varnothing$ (i.e., ${}^2\delta^{\pm}_{2(i)}=\varphi_i(\{x_l\}_{l\in {}^2N_i^{\mp}})$) and from the disjointness of $\beta^{\pm}$ if $b\neq\varnothing$. By (4.13.4),

${}^2\delta^{\pm}_{2(i)} \subset \varphi_{bi}(S^2) \subset \varphi_{bi}(V_i^2)$, and by (4.13.2), $V_i^2 <_{(bi,T_i,m^{n-1})} U_i^2$. Hence we can find in U_i^2 a normal array ${}^2\nu_{2(i)} = \sum_{r=1}^{|{}^2N_i|} \varepsilon(r)\delta_{z_r}$ w.r.t. $\{\varphi_t\}_{t\in T_i^\cdot}$, of order $n-1$ and constant c, so that $\varphi_{bi}(\{z_r\}_{\varepsilon(r)=\pm 1}) = {}^2\delta^{\mp}_{2(i)}$ and, in particular,

$$\varphi_b(\{z_r\}_{\varepsilon(r)=\pm 1}) = {}^2B^{\pm}_{2(i)} \quad \text{and} \quad \varphi_i(\{z_r\}_{\varepsilon(r)=\pm 1}\} = \varphi_i(\{x_l\}_{l\in {}^2N_i^{\mp}}).$$

We take $\{z_r\}_{r=1}^{|{}^2N_i|} = {}^2M_{2(i)}$.

Now we select ${}^3M_{2(i)}$. By (4.16), the elements $l \in {}^3N_i$ have a constant sign. Denote this constant sign by ε (i.e., for $l \in {}^3N_i$, $\varepsilon(l) = \varepsilon$). From the elements of the sequence $\beta^{-\varepsilon}$ which have not been selected yet, we select a sequence ${}^3B_{2(i)}$ of length $|{}^3N_i|$. Since ${}^3B_{2(i)} \subset \varphi_b(V) \subset \varphi_b(W^2)$ and $\varphi_i(\{x_l\}_{l\in {}^3N_i}) \subset \varphi_i(W^2)$ (by (4.13.0)) and since $W^2 \subset_{(bi)} S^2$, we can find in $\varphi_{bi}(S^2) \subset Y_b \times Y_i$ a sequence ${}^3\delta_{2(i)}$, whose projection into Y_b agrees with ${}^3B_{2(i)}$, and whose projection into Y_i agrees with $\varphi_i(\{x_l\}_{l\in {}^3N_i})$. By (4.13.4), ${}^3\delta_{2(i)} \subset \varphi_{bi}(S^2) \subset \varphi_{bi}(V_i^2)$, and as $V_i^2 \subset U_i^2$ (which follows trivially from (4.13.2)) we can find in U_i^2 a sequence ${}^3M_{2(i)}$ of length $|{}^3N_i|$ so that $\varphi_{bi}({}^3M_{2(i)}) = {}^3\delta_{2(i)}$. We also assign a sign to the indices of the sequence ${}^3M_{2(i)}$. The sign of the indices of ${}^3M_{2(i)}$ will be constant, and will be the opposite sign to the sign of ${}^3N_i$, i.e., if the sign of ${}^3N_i$ was ε, then the sign of ${}^3M_{2(i)}$ will be $-\varepsilon$. Note that

$${}^3M_{2(i)} \cap \{z_r : z_r \in {}^2M_{2(i)}, \varepsilon(r) = -\varepsilon({}^3M_{2(i)})\} = \emptyset$$

(where $\varepsilon({}^3M_{2(i)})$ is the constant sign of the indices of ${}^3M_{2(i)}$). Indeed, assume, e.g., that $\varepsilon({}^3M_{2(i)}) = +1$. If $b \neq \emptyset$, then by the construction $\varphi_b({}^3M_{2(i)}) = {}^3B_{2(i)} \subset \beta^+$ while

$$\varphi_b(\{z_r : z_r \in {}^2M_{2(i)}, \varepsilon(r) = -1\}) = {}^2B^-_{2(i)} \subset \beta^-,$$

and since $\beta^+ \cap \beta^- = \emptyset$, we are done. To settle the case when $b = \emptyset$, note that we also have

$$\varphi_i({}^3M_{2(i)}) = \varphi_i(\{x_l\}_{l\in {}^3N_i}) \subset \varphi_i(\{x_l : l \in N_i \setminus {}^1N_i, \varepsilon(l) = -1\}),$$

and by construction of ${}^2M_{2(i)}$,

$$\varphi_i(\{z_r : z_r \in {}^2M_{2(i),\varepsilon(r)=-1}\}) = \varphi_i(\{x_l\}_{l\in {}^2N_i,\varepsilon(l)=+1}).$$

If $b = \emptyset$, then by (4.19) these two sets are disjoint, and our claim follows.

Finally, we construct ${}^1M_{2(i)}$. This is done only if $b = \emptyset$. Applying Lemma 4.1, we select in $U_i^2 \setminus ({}^2M_{2(i)} \cup {}^3M_{2(i)})$ a double array

$${}^1\nu_{2(i)} = \sum_{t=1}^{|{}^1N_i|/2} \varepsilon(t)\delta_{y_t}, \quad {}^1\tilde{\nu}_{2(i)} = \sum_{s=1}^{|{}^1N_i|/2} \tilde{\varepsilon}(s)\delta_{\tilde{y}_s},$$

of order $n-1$, constant c, and norm $\frac{1}{2}|{}^1N_i|$ each, w.r.t. $\{\varphi_r\}_{r\in T_i^*}$ and $\{i\}$. (Note that ${}^1N_i$ has been defined to be the union of disjoint pairs, and hence $\frac{1}{2}|{}^1N_i|$ is an integer; note also that Lemma 4.1 has been formulated for n (and not for $n-1$) and we apply it here for $n-1$, as we can by our induction hypothesis on Theorem 8 for $n-1$. Finally, observe that the "b" from Definition 4.2 of a double array is replaced here by $\{i\}$, i.e., we have $\varphi_i(\{y_t\}_{\varepsilon(t)=\pm1}) = \varphi_i(\{\tilde{y}_s\}_{\tilde{\varepsilon}(s)=\mp1})$.)

We set ${}^1M_{2(i)} = \{y_t\}_{t=1}^{|{}^1N_i|/2} \cup \{\tilde{y}_s\}_{s=1}^{|{}^1N_i|/2}$. Also, we assign signs to the indices t and s, by the corresponding signs in ${}^1\nu_{2(i)}$ and ${}^1\tilde{\nu}_{2(i)}$. $M_{2(i)}$ is defined to be ${}^1M_{2(i)} \cup {}^2M_{2(i)} \cup {}^3M_{2(i)}$, and $M_2 = \bigcup_{i\in T^*} M_{2(i)}$.

REMARK. When we construct $M_{2(i)}$, $i \in T^*$, we may begin with $M_{2(1)}$, then construct $M_{2(2)}$, and so on. In each step, however, we must be careful to select the "new" β's from the ones which have not been selected in earlier steps.

We come now to the inductive step in the construction of the M_j's. Assume that $M_1, M_2, \ldots, M_{j-1}$ have been constructed so that:

(4.20) M_r is a sequence of length m^{n-1}, $\quad 1 \leqq r \leqq j-1$.

(4.21) For each $1 \leqq r \leqq j-1$, M_r is the union $M_r = \bigcup_{i\in T^*} M_{r(i)}$, so that for each $i \in T^*$:

(4.21.1) $$M_{r(i)} \subset U_i^r,$$

and $M_{r(i)}$ is the disjoint union $M_{r(i)} = {}^1M_{r(i)} \cup {}^2M_{r(i)} \cup {}^3M_{r(i)}$, so that

(4.21.2) The points of ${}^2M_{r(i)}$ are the atoms of a normal array ${}^2\nu_{r(i)}$ of order $(n-1)$ and constant c, w.r.t. $\{\varphi_s\}_{s\in T_i^*}$, and the measure ${}^2\nu_{r(j)}$ satisfies ${}^2\nu_{r(i)}(X) = 0$ (i.e., in the sequence ${}^2M_{r(i)}$ there are equally many indices l with $\varepsilon(l) = 1$ and $\varepsilon(l) = -1$).

(4.21.3) If $b \neq \emptyset$ then ${}^1M_{r(i)} = \emptyset$, while if $b = \emptyset$ then the points of ${}^1M_{r(i)}$ are the atoms of a double array ${}^1\nu_{r(i)}$ and ${}^1\tilde{\nu}_{r(i)}$, of order $n-1$ and constant c, w.r.t. $\{\varphi_s\}_{s\in T_i^*}$ and $\{i\}$.

(4.21.4) The sequence ${}^3M_{r(i)}$ is given together with a sign function ε on its set of indices (i.e., $\varepsilon : {}^3M_{r(i)}^* \to \{\pm 1\}$ where ${}^3M_{r(i)}^*$ is the set of indices of ${}^3M_{r(i)}$) so that ε is constant on ${}^3M_{r(i)}^*$ for each $i \in T^*$ and $\Sigma_{i\in T^*} \Sigma_{l\in {}^3M_{r(i)}^*} \varepsilon(l) = 0$.

REMARKS. (i) The index sets of ${}^1M_{r(i)}$, ${}^2M_{r(i)}$ and ${}^3M_{r(i)}$ now have a natural sign function ε: for ${}^3M_{r(i)}$ this follows from (4.21.4), while in the case of ${}^1M_{r(i)}$ and ${}^2M_{r(i)}$ which are atoms of arrays, we adopt the corresponding signs of the arrays.

(ii) Note that the above structure of M_r applies to M_1 too. In this case $M_{r(i)} = \varnothing$ for $1 \neq i \in T^*$, and ${}^1M_{1(1)} = {}^3M_{1(1)} = \varnothing$ too, i.e., $M_1 = {}^2M_{1(1)}$. Actually, we could have introduced the inductive step right after the construction of M_1, but at that point of the construction the introduction of conditions such as (4.21) and its followers could have appeared unnatural to the reader. To avoid this we have constructed M_2 first which, as we hope, explains the sources of (4.21).

Note that from (4.21.2), (4.21.3) and (4.21.4) it follows that

(4.21.5) $$\sum_{l \in M_r^*} \varepsilon(l) = 0.$$

(where $\varepsilon(\cdot)$ is the above-mentioned sign function). Indeed, for a given $i \in T^*$, $\sum_{l \in {}^2M^*_{r(i)}} \varepsilon(l) = 0$ by (4.21.2) and by (4.21.3), $\sum_{l \in {}^1M^*_{r(i)}} \varepsilon(l) = 0$ too. Hence (4.21.5) follows from (4.21.4).

As in the construction of M_2, before constructing M_j, we introduce a reordering of the index set M^*_{j-1} of M_{j-1}. Actually, we shall reorder M^*_r for all $r \leqq j-1$. We begin with the following

(4.22) CLAIM. *There exists a function $\tau : \bigcup_{i \in T^*} {}^3M^*_{r(i)} \to T^*$ (where ${}^3M^*_{r(i)}$ is the set of indices of ${}^3M_{r(i)}$) such that for $l \in {}^3M^*_{r(i)}$, $\tau(l) \neq i$, and so that for all $i \in T^*$, $\sum_{l \in \tau^{-1}(i)} \varepsilon(l) = 0$.*

Indeed, set $d = \min_{i \in T^*} |{}^3M^*_{r(i)}|$. Let us assume that $d = |{}^3M^*_{r(1)}|$ and that ε attains different values on ${}^3M^*_{r(1)}$ and ${}^3M^*_{r(2)}$. (There is no loss of generality in these assumptions.)

Let $G \subset {}^3M^*_{r(2)}$ be a set so that $d = |G|$. (G exists by the minimality of $|{}^3M^*_{r(1)}|$.) Define now τ on $\bigcup_{i \in T^*} {}^3M^*_{r(i)}$ by

$$\tau(l) = \begin{cases} 3 & \text{if } l \in {}^3M^*_{r(1)} \cup G, \\ 1 & \text{if } l \in {}^3M^*_{r(2)} \setminus G \text{ or } l \in \bigcup\limits_{\substack{i \in T^* \\ i \geqq 3}} {}^3M^*_{r(i)}. \end{cases}$$

(Recall that $T^* = \{1, 2, \dots, |T^*|\}$ and that $|T^*| \geqq 4$ since $n \geqq 2$.)

It follows at once from this definition that if $l \in {}^3M^*_{r(i)}$ then $\tau(l) \neq i$. Also, if $i \notin \{1, 3\}$ then $\tau^{-1}(i) = \varnothing$. For $i = 3$, $\tau^{-1}(i) = {}^3M^*_{r(1)} \cup G$; hence

$$\sum_{l \in \tau^{-1}(3)} \varepsilon(l) = \sum_{l \in {}^3M^*_{r(i)}} \varepsilon(l) + \sum_{l \in G} \varepsilon(l),$$

and since both ${}^3M^*_{r(1)}$ and G have the same cardinality d, and ε attains opposite signs on these sets, the sum is 0. Finally,

$$\sum_{l\in\tau^{-1}(1)}\varepsilon(l)=\sum_{l\in\bigcup_{i\in T^*}{}^3M^*_{r(i)}}\varepsilon(l)-\sum_{l\in\tau^{-1}(3)}\varepsilon(l),$$

and both terms in this sum are 0, the first by (4.21.4) and the second by the above observation. This proves (4.22).

We now wish to extend the function τ of (4.22) to M^*_r. This is done as follows. Let $\tilde{\tau}:\{\omega:\omega\subset T^*, |\omega|\leqq 2n-1\}\to T^*$ be a function so that

$$\tilde{\tau}(\omega)\not\in\omega. \tag{4.23}$$

This is possible since $|T^*|\geqq 2n$. Let $l\in M^*_r\setminus\bigcup_{i\in T^*}{}^3M^*_{r(i)}$. Then l is an index of an atom x_l in one (and only one) of the arrays ${}^2\nu_{r(i)}$, ${}^1\nu_{r(i)}$ or ${}^1\tilde{\nu}_{r(i)}$, $i\in T^*$ (by (4.21)) and, as such, the set $\sigma(l)$ is well defined, so that $|\sigma(l)|\leqq 2(n-1)$ and $\sigma(l)\subset T^*_i$ (by (4.21) and (ar.3.2)). We now define

$$\tau(l)=\tilde{\tau}(\{i\}\cup\sigma(l)),\qquad l\in M^*_r\setminus\bigcup_{i\in T^*}{}^3M^*_{r(i)}. \tag{4.24}$$

Thus, by (4.22) and (4.24) τ is now a well-defined function from M^*_r into T^*. Note that by (4.22), (4.23) and (4.24) the following holds:

(4.25) For $l\in M^*_r$, if $l\in M^*_{r(i)}$ then $\tau(l)\neq i$, and if l is an index of an atom of an array in $M_{r(i)}$ (i.e., $l\in{}^1M_{r(i)}\cup{}^2M_{r(i)}$) then also $\tau(l)\not\in\{i\}\cup\sigma(l)$.

Set

$$N_{r(i)}=\tau^{-1}(i),\qquad i\in T^*. \tag{4.26}$$

(4.27) CLAIM. *For all* $i_0\in T^*$, $|\sum_{l\in N_{r(i_0)}}\varepsilon(l)|\leqq 3|T^*|2^{|T^*|}cm^{n-2}$.

Indeed,

$$N_{r(i_0)}=\left(N_{r(i_0)}\cap\left(\bigcup_{i\in T^*}{}^3M^*_{r(i)}\right)\right)\cup\bigcup_{i\in T^*}((N_{r(i_0)}\cap{}^2M^*_{r(i)})\cup(N_{r(i_0)}\cap{}^1M^*_{r(i)}))$$

and this is a disjoint union. We shall estimate $\sum\varepsilon(l)$ on each of these sets. So, let $i_0\in T^*$ be fixed.

(i) By (4.22),

$$\sum_{l\in N_{r(i_0)}\cap(\bigcup_{i\in T^*}{}^3M^*_{r(i)})}\varepsilon(l)=0.$$

(ii) ${}^2M^*_{r(i)}$ is the index set of the atoms ${}^2M_{r(i)}$ of the *normal* array ${}^2\nu_{r(i)}$ of order $(n-1)$ with the constant c, and as ${}^2M_{r(i)} \subset M_r$, it follows from (4.20) that

$$\|{}^2\nu_{r(i)}\| = |{}^2M^*_{r(i)}| \leqq m^{n-1}.$$

Thus, by (4.26), (3.24) and (ar. N),

$$\Big|\sum_{l \in N_{r(i_0)} \cap {}^2M^*_{r(i)}} \varepsilon(l)\Big| = \Big|\sum_{l \in {}^2M^*_{r(i)}, \bar{\tau}(\{i\}\cup\sigma(l)) = i_0} \varepsilon(l)\Big| \leqq \sum_{\substack{\sigma \subset T^*_i \\ |\sigma| \leqq 2(n-1) \\ \bar{\tau}(\{i\}\cup\sigma) = i_0}} \Big|\sum_{\substack{l \in {}^2M^*_{r(i)} \\ \sigma(l) = \sigma}} \varepsilon(l)\Big|$$

$$\leqq 2^{|T^*_i|} \Big|\sum_{\substack{l \in {}^2M^*_{r(i)} \\ \sigma(l) = \sigma}} \varepsilon(l)\Big| \leqq 2^{|T^*_i|} c m^{n-2} < 2^{|T^*|} c m^{n-2}.$$

(iii) ${}^1M^*_{r(i)}$ is the union of two sets of indices, each of which is the set of indices of atoms of some normal array (${}^1\nu_{r(i)}$ and ${}^1\tilde{\nu}_{r(i)}$) of order $(n-1)$, constant c, and norm $\leqq m^{n-1}$. Hence, an estimate as in (ii) applies to each of these sets, and it follows that

$$\Big|\sum_{l \in N_{r(i)} \cap {}^1M^*_{r(i)}} \varepsilon(l)\Big| \leqq 2 \cdot 2^{|T^*|} c m^{n-2}.$$

(4.27) now follows from (i), (ii) and (iii). (Note that the estimates in (4.27) are very generous, and can easily be improved; however, we find it convenient to use this estimate. The main point in (4.27) is that the bound for $|\sum_{l \in N_{r(i)}} \varepsilon(l)|$ does not depend on r.)

Now we decompose each $N_{r(i)}$ into 3 sets ${}^1N_{r(i)}$, ${}^2N_{r(i)}$ and ${}^3N_{r(i)}$, in the very same way we decomposed N_i before constructing M_2.

(4.28) *Selection of* ${}^1N_{r(i)}$. If $b \neq \emptyset$ we take ${}^1N_{r(i)} = \emptyset$. If $b = \emptyset$, we select a maximal number of disjoint pairs $\{l, l'\} \subset N_{r(i)}$, with $\varepsilon(l) \cdot \varepsilon(l') = -1$ and $\varphi_i(x_l) = \varphi_i(x_{l'})$, and let ${}^1N_{r(i)}$ be the union of these pairs.

(Recall that x_l is the point in M_r whose index is $l \in N_{r(i)}$.) Note that

(4.29) If $b = \emptyset$, then for l, l' in $M_{r(i)} \setminus {}^1N_{r(i)}$, $\varepsilon(l) \cdot \varepsilon(l') = -1$ implies that $\varphi_i(x_l) \neq \varphi_i(x_{l'})$.

From (4.28) it also follows that

(4.30) $$\sum_{l \in {}^1N_{r(i)}} \varepsilon(l) = 0.$$

(4.31) *Selection of* ${}^2N_{r(i)}$. ${}^2N_{r(i)}$ is selected to be a subset of $N_{r(i)}\setminus{}^1N_{r(i)}$, with maximal cardinality, so that $\sum_{l\in{}^2N_{r(i)}}\varepsilon(l)=0$ (cf. the corresponding selection of ${}^2N_i$).

(4.32) *Selection of* ${}^3N_{r(i)}$. ${}^3N_{r(i)}=N_{r(i)}\setminus({}^1N_{r(i)}\cup{}^2N_{r(i)})$.

We claim that

(4.33) The sign function ε is constant on ${}^3N_{r(i)}$ for each i, and $\sum_{i\in T^*}\sum_{l\in{}^3N_{r(i)}}\varepsilon(l)=0$

and also

(4.34) $$|{}^3N_{r(i)}|\leqq 3|T^*|2^{|T^*|}cm^{n-2}.$$

Indeed, if l and l' are in ${}^3N_{r(i)}$ and $\varepsilon(l)\cdot\varepsilon(l')=-1$, then we can add both l and l' to ${}^2N_{r(i)}$ without violating (4.31), and this contradicts the maximality of ${}^2N_{r(i)}$. Hence ε is constant on ${}^3N_{r(i)}$. Also,

$$\sum_{i\in T^*}\sum_{l\in{}^3N_{r(i)}}\varepsilon(l)=\sum_{i\in T^*}\sum_{l\in N_{r(i)}}\varepsilon(l)-\left(\sum_{i\in T^*}\sum_{l\in{}^1N_{r(i)}}\varepsilon(l)+\sum_{i\in T^*}\sum_{l\in{}^2N_{r(i)}}\varepsilon(l)\right).$$

The first term in this sum is 0 by (4.21.5) (since $\bigcup_{i\in T^*}N_{r(i)}=M^*_{r(i)}$). The other terms vanish by (4.30) and (4.31). This proves (4.33). Hence,

$$|{}^3N_{r(i)}|=\left|\sum_{l\in{}^3N_{r(i)}}\varepsilon(l)\right|=\left|\sum_{l\in N_{r(i)}}\varepsilon(l)\right|$$

(since the sums over ${}^1N_{r(i)}$ and ${}^2N_{r(i)}$ vanish by (4.30) and (4.31) and

$$\left|\sum_{l\in N_{r(i)}}\varepsilon(l)\right|\leqq 3|T^*|2^{|T^*|}cm^{n-2}$$

by (4.27), which proves (4.34).

For a subset P of M^*_r let $P^{\pm}=\{l\in P:\varepsilon(l)=\pm1\}$. Finally we have

(4.35) If $b=\varnothing$ then

$$\varphi_i(\{x_l\}:l\in(N_{r(i)}\setminus{}^1N_{r(i)})^+)\cap\varphi_i(\{x_l\}:l\in(N_{r(i)}\setminus{}^1N_{r(i)})^-)=\varnothing.$$

This follows from the maximality of ${}^1N_{r(i)}$ (cf. (4.28)). In particular we have

$$\varphi_i(\{x_l\}:l\in{}^2N^+_{r(i)})\cap\varphi_i(\{x_l\}:l\in{}^2N^-_{r(i)})=\varnothing.$$

We are now ready to construct M_j. M_j will be constructed to be a sequence which satisfies (4.20), (4.21) and (4.21.p), $1\leqq p\leqq 4$. The construction is practically identical to the construction of M_2. Hence are the details.

(4.36) *Construction of* ${}^2M_{j(i)}$. By (4.31)

$$|{}^2N^+_{(j-1)(i)}| = |{}^2N^-_{(j-1)(i)}| = \tfrac{1}{2}|{}^2N_{(j-1)(i)}|.$$

Let ${}^2B^+_{j(i)}$ and ${}^2B^-_{j(i)}$ be subsequences of β^+ and β^- respectively, of length $\frac{1}{2}|{}^2N_{(j-1)(i)}|$ each, which consists of elements of β^+ and β^- that have not been selected earlier in the construction. Then ${}^2B^\pm_{j(i)} \subset \varphi_b(V) \subset \varphi_b(W^j)$ (by (4.13.0)) and since ${}^2N_{(j-1)(i)} \subset M_{j-1} \subset \bigcup_{i\in T^*} U_i^{j-1} \subset W^j$ ((4.21.1) and (4.13.0)), and $W^j \subset_{(bi)} S^j$ (by (4.13.3)) we can find (by (4.2)) in $\varphi_{bi}(S^j)$ two sequences ${}^2\delta^+_{j(i)}$ and ${}^2\delta^-_{j(i)}$, of length $\frac{1}{2}|{}^2N_{(j-1)(i)}|$ each, so that the projection of ${}^2\delta^\pm_{j(i)}$ into Y_b is ${}^2B^\pm_{j(i)}$, while the projection of ${}^2\delta^+_{j(i)}$ into Y_i is $\varphi_i(\{x_l\}: l \in {}^2N^-_{(j-1)(i)})$ and the projection of ${}^2\delta^-_{j(i)}$ into Y_i is $\varphi_i(\{x_l\}: l \in {}^2N^+_{(j-1)(i)})$. ${}^2\delta^+_{j(i)} \cap {}^2\delta^-_{j(i)} = \emptyset$. This follows from (4.35) if $b = \emptyset$, and from the fact that $\beta^+ \cap \beta^- = \emptyset$, if $b \neq \emptyset$. By (4.13.4) ${}^2\delta^\pm_{j(i)} \subset \varphi_{bi}(S^j) \subset \varphi_{bi}(V_i^j)$, and by (4.13.2) $V_i^j <_{(bi,T_i,m^{n-1})} U_i^j$. Hence there exists in U_i^j a normal array

$${}^2\nu_{j(i)} = \sum_{s=1}^{|{}^2N_{(j-1)(i)}|} \varepsilon(s)\delta_{z_s}$$

w.r.t. $\{\varphi_t\}_{t\in T_i^*}$, of order $n-1$, constant c, and norm $|{}^2N_{(j-1)(i)}|$, so that

$$\varphi_{bi}(\{z_s\}: \varepsilon(s) = \pm 1) = {}^2\delta^\pm_{j(i)},$$

and in particular $\varphi_b(\{z_s\}: \varepsilon(s) = \pm 1) = {}^2B^\pm_{j(i)}$ while

$$\varphi_i(\{z_s\}: \varepsilon(s) = \pm 1) = \varphi_i(\{x_l\}: l \in {}^2N^\mp_{(j-1)(i)}).$$

(Note the $\pm$ and $\mp$!) We set

$${}^2M_{j(i)} = \{z_s\}, \qquad 1 \leqq s \leqq |{}^2N_{(j-1)(i)}|.$$

(4.37) *Construction of* ${}^3M_{j(i)}$. By (4.33) the sign function ε is constant on ${}^3N_{(j-1)(i)}$. Denote this constant by ε. Select a subsequence ${}^3B_{j(i)}$ of $\beta^{-\varepsilon} \setminus \{$the elements of $\beta^{-\varepsilon}$ which have already been selected$\}$ of length $|{}^3N_{(j-1)(i)}|$. Since ${}^3B_{j(i)} \subset \varphi_b(V) \subset \varphi_b(W^j)$ and $\varphi_i(\{x_l\}: l \in {}^3N_{(j-1)(i)}) \subset \varphi_i(W^j)$ (by (4.13.0)) and since $W^j \subset_{(bi)} S^j$, we can find in $\varphi_{bi}(S^j)$ a sequence ${}^2\delta_{j(i)}$ whose projection into Y_b agrees with ${}^3B_{j(i)}$ while its projection into Y_i is $\varphi_i(\{x_l\}: l \in {}^3N_{(j-1)(i)})$. By (4.13.4) ${}^3\delta_{j(i)} \subset \varphi_{bi}(S^j) \subset \varphi_{bi}(V_i^j)$, and since from (4.13.2) it follows that $V_i^j \subset U_i^j$, we can find in U_i^j a sequence

${}^3M_{j(i)}$ of length $|{}^3N_{(j-1)(i)}|$, so that $\varphi_{bi}({}^3M_{j(i)}) = {}^3\delta_{j(i)}$. We also assign a sign function ε to the indices ${}^3M^*_{j(i)}$ of ${}^3M_{j(i)}$, which will be constant on ${}^3M^*_{j(i)}$, and will be the opposite sign to the sign of ${}^3N_{(j-1)(i)}$ (i.e., if the sign of ${}^3N_{(j-1)(i)}$ is ε, the sign of ${}^3M_{j(i)}$ is $-\varepsilon$).

We claim that

(4.38) $${}^3M_{j(i)} \cap \{z_s\,;\, z_s \in {}^2M_{j(i)}, \varepsilon(s) = -\varepsilon({}^3M^*_{j(i)})\} = \emptyset$$

(where $\varepsilon({}^3M_{j(i)})$ is the constant sign of this set).

Indeed, assume, e.g., that $\varepsilon({}^3M^*_{j(i)}) = 1$. If $b \neq \emptyset$ then $\varphi_b({}^3M_{j(i)}) = {}^3B_{j(i)} \subset \beta^+$, while

$$\varphi_b(\{z_r : \varepsilon(r) = -1\} = {}^2B^-_{j(i)} \subset \beta^-$$

by (4.36), and since $\beta^+ \cap \beta^- = \emptyset$, (4.38) follows. If $b = \emptyset$, we argue as follows: by (4.37) and (4.32)

$$\varphi_i({}^3M_{j(i)}) = \varphi_i(\{x_l\} : l \in {}^3N_{(j-1)(i)}) \subset \varphi_i(\{x_l\} : l \in (N_{(j-1)(i)} \setminus {}^1N_{(j-1)(i)})^-),$$

and by (4.36)

$$\varphi_i(\{z_s : z_s \in {}^2M_{j(i)}, \varepsilon(s) = -1\}) = \varphi_i(\{x_l\} : l \in {}^2N^+_{(j-1)(i)}).$$

Hence, if $b = \emptyset$, these two sets are disjoint by (4.35).

(4.39) *Construction of* ${}^1M_{j(i)}$ (only if $b = \emptyset$). Applying Lemma 4.1 we select in $U^j_i \setminus ({}^2M_{j(i)} \cup {}^3M_{j(i)})$ a double array

$${}^1\nu_{j(i)} = \sum_{s=1}^{|{}^1N_{(j-1)(i)}|/2} \varepsilon(s)\delta_{y_s} \quad \text{and} \quad {}^1\tilde{\nu}_{j(i)} = \sum_{s=1}^{|{}^1N_{(j-1)(i)}|/2} \tilde{\varepsilon}(s)\delta_{\tilde{y}_s},$$

of order $n-1$, constant c and norm $\frac{1}{2}|{}^1N_{(j-1)(i)}|$ each, w.r.t. $\{\varphi_\iota\}_{\iota \in T'_i}$ and $\{i\}$. We set

$${}^1M_{j(i)} = \{y_s\} \cup \{\tilde{y}_s\}, \qquad 1 \leqq s \leqq \tfrac{1}{2}|{}^1N_{(j-1)(i)}|.$$

Now we define

(4.40) $$M_{j(i)} = {}^1M_{j(i)} \cup {}^2M_{j(i)} \cup {}^3M_{j(i)} \quad \text{and} \quad M_j = \bigcup_{i \in T^*} M_{j(i)}.$$

By (4.39), (4.37) and (4.36), $M_{j(i)} \subset U^j_i$. Also, from the above and (4.38) it follows that if x_l and $x_{l'}$ are in $M_{j(i)}$, and $\varepsilon(l) \cdot \varepsilon(l') = -1$, then $x_l \neq x_{l'}$. Since the sets U^j_i are disjoint (by (4.13.1)), $\varepsilon(l)\varepsilon(l') = -1$ implies $x_l \neq x_{l'}$ for x_l, $x_{l'}$ in M_j.

We check now that (4.20), (4.21) and (4.21, p), $1 \leqq p \leqq 4$ hold for M_j.

By (4.39), (4.37) and (4.36) the sequence ${}^pM_{j(i)}$, $p = 1,2,3$ is of the same length as the set ${}^pN_{(j-1)(i)}$. By (4.26), $\bigcup_{i\in T^*} N_{(j-1)(i)}$ is a decomposition of $M_{(j-1)}$, and $N_{(j-1)(i)} = \bigcup_{p=1}^{3} {}^pN_{(j-1)(i)}$ is a decomposition too (by (4.28), (4.31) and (4.32)). Thus, since by (4.20) $|M^*_{(j-1)}| = m^{n-1}$, the same holds for M^*_j too.

(4.21) and (4.21.1) for M_j follow from (4.40), (4.36), (4.37) and (4.38).

(4.21.2) for M_j follows from (4.36), and (4.21.3) from (4.39). To verify (4.21.4) for M_j, note that by (4.37)

$$\varepsilon({}^3M_{j(i)}) = -\varepsilon({}^3N_{(j-1)(i)}) \qquad \text{for all } i \in T^*.$$

Hence (4.21.4) for M_j follows from (4.33).

This completes the inductive construction of M_j, $1 \leqq j \leqq m$. Note that, by (4.20), $|\bigcup_{j=1}^{m} M^*_j| = m \cdot m^{n-1} = m^n$. (We assume here, as we clearly may, that the index sets for different M_j's are disjoint.)

We still have to construct M_{n+1}. This is done as follows.

(4.41) *Construction of* M_{m+1}. By (4.21.2), (4.21.3) and (4.21.4)

$$|M^{*+}_j| = |M^{*-}_j| = \tfrac{1}{2}m^{n-1} \qquad \text{for all } 2 \leqq j \leqq m,$$

and by the construction of M_1, the same holds for M_1 too. So, if $b \neq \emptyset$, then by (4.36) and (4.37) $\varphi_b(\{x_l : x_l \in \bigcup_{j=1}^{m} M_j,\ \varepsilon(l) = \pm 1\})$ is a subsequence of $\beta^{\pm}$ of length $\frac{1}{2}m^n$. Recall that $|\beta^+| + |\beta^-| = L$ and $||\beta^+| - |\beta^-|| \leqq 1$. (Note that $|\beta|$ for a sequence β denotes its length, and not its cardinality as a set.) m was chosen to be the largest even integer with $m^n \leqq L$. So, if $m^n < L$, Let ${}^1\beta^{\pm}$ be the subsequence of $\beta^{\pm}$ which remains after removing from $\beta^{\pm}$ the subsequences $\varphi_b(\{x_l \in \bigcup_{j=1}^{m} M_j, \varepsilon(l) = \pm 1\})$. Let M_{m+1} be the union of two sequences M^+_{m+1} and M^-_{m+1} in V so that $\varphi_b(M^{\pm}_{m+1}) = {}^1\beta^{\pm}$. We also extend the sign function ε to $M = \bigcup_{j=1}^{m+1} M_j$, by letting ε be $+1$ on the indices of M^+_{m+1} and -1 on the indices of M^-_{m+1}. (Note that $M^+_{m+1} \cap M^-_{m+1} = \emptyset$ since $\beta^+ \cap \beta^- = \emptyset$. Also $M_{m+1} \cap M_j = \emptyset$ for $1 \leqq j \leqq m$, since $M_{m+1} \subset V \subset S^1$, while $M_j \subset \bigcup_{i\in T^*} U^j_i$, and $S^1 \cap U^j_i = \emptyset$ for all $1 \leqq j \leqq m$ and $i \in T^*$.) If $b = \emptyset$, then we select in V any sequence M_{m+1} of length $L - m^n$, which consists of different elements, and extend the sign function ε to its indices arbitrarily. Note that in this case too M_{n+1} does not meet $\bigcup_{j=1}^{m} M_j$. Also, in both cases

$$|M_{m+1}| = L - m^n \leqq 2^{|T^*|} L^{(n-1)/n}$$

by (4.8).

This completes the construction of $M = \bigcup_{j=1}^{m+1} M_j$. Recall that each M_j has been constructed as a union of sequences. Let us assume that the indices of those sequences are disjoint, and are so ordered that M itself becomes the sequence $M = \{x_l\}_{l=1}^{L}$ under the same indexing. Set

$$\mu = \sum_{l=1}^{L} \varepsilon(l)\delta_{x_l}. \tag{4.42}$$

We shall show that μ is a normal array, of order n, constant

$$c(n, |T^*|) = 9|T^*|^2 2^{|T^*|} c$$

and norm L, w.r.t. $\{\varphi_i\}_{i\in T^*}$, such that $\varphi_b(\{x_l\}_{\varepsilon(l)=\pm1}) = \beta^{\pm}$, and $\{x_l\}_{l=1}^{L} \subset U$, which will prove Theorem 8. (Clearly, the signs $\varepsilon(l)$ are the ones assigned to the indices $1 \leqq l \leqq L$ through the construction.)

Note first that, by (4.21.1) and (4.41), $M = \{x_l\}_{1\leqq l\leqq L} \subset U$. Also, by (4.41), $\varphi_b(\{x_l\}_{\varepsilon(l)=\pm1}) = \beta^{\pm}$. If $b \neq \emptyset$ this also implies that for $\varepsilon(l)\varepsilon(l') = -1$, $x_l \neq x_{l'}$ (since $\beta^+ \cap \beta^- = \emptyset$). If $b = \emptyset$ the above still holds. Indeed, as the sets $M_{j,1\leqq j\leqq m+1}$ are mutually disjoint (by (4.21) and (4.21.1)) we may check each of them separately. If x_l and $x_{l'} \in M_{m+1}$, $\varepsilon(l)\cdot\varepsilon(l') = -1$ implies $x_l \neq x_{l'}$ by (4.41), while if $x_l, x_{l'} \in M_j$, for some $1 \leqq j \leqq m$, this follows from the statement after (4.40). This shows that μ satisfies (ar.2), i.e., $\|\mu\| = L$.

To check (ar.3), we have to define the subsets L_i of $\{1, 2, \dots, L\}$, $i \in T^*$. So fix some $i_0 \in T^*$. For $1 \leqq j \leqq m$, $M_j = \bigcup_{i\in T^*} M_{j(i)}$, and

$$M_{j(i)} = {}^1M_{j(i)} \cup {}^2M_{j(i)} \cup {}^3M_{j(i)}$$

((4.21) and (4.21.1)). For $i \neq i_0$, ${}^2M_{j(i)}$ is a sequence whose elements are the atoms of the normal array ${}^2\nu_{j(i)}$ of order $n-1$ w.r.t. $\{\varphi_t\}_{t\in T_i^*}$ (by (4.21.2)). If $i_0 \in T_i^*$, let $L_{i_0}^{2,j,i}$ denote the subset of ${}^2M_{j(i)}^*$, which corresponds to ${}^2\nu_{j(i)}$ by (ar.3). By (4.21.3), ${}^1M_{j(i)}$ consists of the atoms of the double array ${}^1\nu_{j(i)}$ and ${}^1\tilde{\nu}_{j(i)}$, of order $n-1$, w.r.t. $\{\varphi_t\}_{t\in T_i^*}$, and i. Again, if $i_0 \in T_i^*$, let $L_{i_0}^{1,j,i}$ and $\tilde{L}_{i_0}^{1,j,i}$ be the subsets of the index sets of ${}^1\nu_{j(i)}$ and ${}^1\tilde{\nu}_{j(i)}$ which are guaranteed by (ar.3).

Note that by (ar.3.1) and (ar.3.1′) we have

(4.43) $(\Sigma\,\varepsilon(l)\delta_{x_l})\circ\varphi_{i_0}^{-1} = 0$, where the summation is taken over $l \in L_{i_0}^{2,j,i}$, $l \in L_{i_0}^{1,j,i}$, or $l \in \tilde{L}_{i_0}^{1,j,i}$. Or, equivalently, there exist decompositions $E_{i_0}^{2,j,i}$, $E_{i_0}^{1,j,i}$ and $\tilde{E}_{i_0}^{1,j,i}$ of $L_{i_0}^{2,j,i}$, $L_{i_0}^{1,j,i}$ and $\tilde{L}_{i_0}^{1,j,i}$ respectively, each of which consists of disjoint pairs $\{l, l'\}$ of indices, such that $\varepsilon(l)\cdot\varepsilon(l') = -1$ and $\varphi_{i_0}(x_l) = \varphi_{i_0}(x_{l'})$.

We can now define L_{i_0}. The selection of indices in L_{i_0} will be so organized that they will appear in disjoint pairs $\{l, l'\}$ with $\varepsilon(l)\cdot\varepsilon(l') = -1$ and $\varphi_{i_0}(x_l) = \varphi_{i_0}(x_{l'})$.

(4.44) The sets $L_{i_0}^{2,j,i}$, $L_{i_0}^{1,j,i}$ and $\tilde{L}_{i_0}^{1,j,i}$, $1 \leqq j \leqq m$, $i \in T^*$, $i \neq i_0$ will all be contained in L_{i_0}. Note that the pairings $E_{i_0}^{2,j,i}$, $E_{i_0}^{1,j,i}$ and $\tilde{E}_{i_0}^{1,j,i}$ of these sets (from (4.43)) induce a disjoint pairing on their union, since the sets themselves are disjoint, and each pairing consists of disjoint pairs by (4.43).

In addition, L_{i_0} will consist of the following indices.

(4.44.1) If $x_l \in {}^1M_{j(i_0)}$, i.e., x_l is an atom of either ${}^1\nu_{j(i_0)}$ or ${}^1\tilde{\nu}_{j(i_0)}$, then in each case there exists, by Definition (4.2) of a double array w.r.t. $\{\varphi_i\}_{i \in T_{i_0}^*}$ and i_0 (and (4.6) in particular), some atom $x_{l'}$ of the other array, with $\varepsilon(l) \cdot \varepsilon(l') = -1$ and $\varphi_{i_0}(x_l) = \varphi_{i_0}(x_{l'})$. Let $D_{i_0}^j$ denote the collection of all these pairs $\{l, l'\}$. Note that their union is ${}^1M_{j(i_0)}^*$. So we add ${}^1M_{j(i_0)}^*$ to L_{i_0}. Clearly, the pairs in $D_{i_0}^j$ are mutually disjoint, and by (4.21.1) they are also distinct from the pairs which have already been selected.

(4.44.2) If $x_l \in {}^2M_{j(i_0)}$ for some $2 \leqq j \leqq m$, then by (4.36) there exists some $l' \in {}^2N_{j-1,(i_0)}$ so that $\varepsilon(l)\varepsilon(l') = -1$ and $\varphi_{i_0}(x_l) = \varphi_{i_0}(x_{l'})$, where $x_{l'} \in M_{j-1(i_0)}$ is the element with index l' (cf. the statement preceding (4.37)). Let $G_{i_0}^j$ denote the collection of all these pairs. We add the union of $G_{i_0}^j$ (i.e., ${}^2M_{j(i_0)}^* \cup {}^2N_{j-1(i_0)}$) to L_{i_0}. Note that this set is disjoint from the ones selected earlier; indeed, for the elements of ${}^2M_{j(i_0)}^*$ this follows from (4.21.1), while for $l \in {}^2N_{j-1(i_0)}$ (actually for $l \in N_{j-1(i_0)}$) we have: l is an element of M_{j-1}^*, and thus $l \in M_{j-1(i)}^*$ for some $i \in T^*$. By (4.25) and (4.26) we have that $i_0 = \tau(l) \notin \{i\} \cup \sigma(l)$ (cf. also (4.23) and (4.24)). Thus, l cannot be an element of the sets that have been assigned earlier to L_{i_0}, since by (4.44) and (4.44.1), for each l in one of these sets, either $i_0 \in \sigma(l)$ (for l in $L_{i_0}^{2,j-1,i}$, $L_{i_0}^{1,j-1,i}$ and $\tilde{L}_{i_0}^{1,j-1,i}$) or $i = i_0$ (for ${}^1M_{j-1(i_0)}^*$).

(4.44.3) If $x_l \in {}^3M_{j(i_0)}$, for some $2 \leqq j \leqq m$, then by (4.37) there exists some $l' \in {}^3N_{j-1(i_0)}$ with $\varepsilon(l)\varepsilon(l') = -1$ and $\varphi_{i_0}(x_l) = \varphi_{i_0}(x_{l'})$, where $x_{l'} \in M_{j-1}$ is the element with index l'. Let $H_{i_0}^j$ denote the collection of all those pairs $\{l, l'\}$. Their union ${}^3M_{j(i_0)}^* \cup {}^3N_{j-1(i_0)}$ is added to L_{i0}. Clearly, $H_{i_0}^j$ consists of disjoint pairs, and the same argument as in (4.44.2) shows that ${}^3M_{j(i_0)}^* \cup {}^3N_{j-1(i_0)}$ does not contain any of the indices which have been selected earlier.

(4.44.4) Finally, if $l \in {}^1N_{j(i_0)}$ for some $1 \leqq j \leqq m$, then by (4.28) there exists some $l' \in {}^1N_{j(i_0)}$ with $\varepsilon(l)\varepsilon(l') = -1$ and $\varphi_{i_0}(x_l) = \varphi_{i_0}(x_{l'})$. Let $I^j_{i_0}$ denote the collection of all these pairs. We add its union ${}^1N_{j(i_0)}$ to L_{i_0}. As in (4.44.2) and (4.44.3), the definition (4.26) of $N_{j(i_0)}$, (4.25) and (4.24) show that the elements of ${}^1N_{j(i_0)}$ have not been selected earlier in the construction of L_i.

This completes the selection of L_{i_0}. It follows from (4.44) and (4.44.r), $1 \leqq r \leqq 4$, that the pairing of L_{i_0} induced by $E^{2,j,i}_{i_0}$, $E^{1,j,i}_{i_0}$, $\tilde{E}^{1,j,i}_{i_0}$, $G^j_{i_0}$, $D^j_{i_0}$, $H^j_{i_0}$ and $I^j_{i_0}$ is disjoint, and also satisfies $\varepsilon(l)\varepsilon(l') = -1$ and $\varphi_{i_0}(x_l) = \varphi_{i_0}(x_{l'})$ for every pair $\{l, l'\}$ in this deomposition, and (ar.3.1) follows.

We still have to check (ar.3.2) and (ar.N).

For $1 \leqq l \leqq L$ let

$$\xi(l) = \{i : i \in T^*, l \in L_i\}. \tag{4.45}$$

(We shall preserve the letter σ for the corresponding sets in the arrays ${}^2\nu_{j(i)}$, ${}^1\nu_{j(i)}$ and ${}^1\tilde{\nu}_{j(i)}$; as in (4.24), we shall also make the following convention: if $l \in \bigcup_{i \in T^*} {}^3M^*_{j(i)}$ (i.e., l is not an index of an atom in some array of order $n-1$) then we put $\sigma(l) = \varnothing$.)

We claim that the following holds:

(4.46) Let $1 \leqq l \leqq L$. Then $\sigma(l) \subset \xi(l)$, $|\xi(l)| \leqq |\sigma(l)| + 2$, and if $l \in M^*_j$ for some $2 \leqq j \leqq m-1$ then $|\xi(l)| = |\sigma(l)| + 2$.

Indeed, let $i_0 \in \sigma(l)$. The corresponding x_l is then an atom of some ${}^2\nu_{j(i)}$, ${}^1\nu_{j(i)}$ or ${}^1\tilde{\nu}_{j(i)}$, with $i \neq i_0$ (since those are arrays w.r.t. $\{\varphi_t\}_{t \in T^*_i}$ and $i \notin T^*_i$). Then by the definition of $L^{2,j,i}_{i_0}$, $L^{1,j,i}_{i_0}$ and $\tilde{L}^{1,j,i}_{i_0}$, l will be an element of one of these sets, and by (4.44) $l \in L_{i_0}$, i.e., $i_0 \in \xi(l)$. Hence $\sigma(l) \subset \xi(l)$. From the above it also follows that if $i_0 \in \xi(l) \setminus \sigma(l)$, then l must be an element of one of the pairs in $D^j_{i_0}$, $G^j_{i_0}$, $H^j_{i_0}$, or $I^j_{i_0}$. But this can occur for at most 2 values of i_0. Indeed, assume, e.g., that $l \in M^*_j$ for some $1 \leqq j \leqq m$. In (4.44.r), $1 \leqq r \leqq 4$, we have defined $D^j_{i_0}$, $G^j_{i_0}$, $H^j_{i_0}$, and $I^j_{i_0}$ as pairs $\{l, l'\}$. In order to be an element in one of these pairs, l must satisfy either $l \in {}^1M^*_{j(i_0)} \cup {}^2M^*_{j(i_0)} \cup {}^3M^*_{j(i_0)} \cup {}^1N_{j(i_0)}$ or $l \in {}^2N_{j(i_0)} \cup {}^3N_{j(i_0)}$. (The second possibility follows from (4.44.2) and (4.44.3), when l is actually the "l'" in the construction of $G^{j+1}_{i_0}$ and $H^{j+1}_{i_0}$.) But ${}^1M^*_{j(i_0)}$, ${}^2M^*_{j(i_0)}$ and ${}^3M^*_{j(i_0)}$ are mutually disjoint (by (4.21.1)) and such are the sets ${}^1N_{j(i_0)}$, ${}^2N^*_{j(i_0)}$ and ${}^3N_{j(i_0)}$ too (by (4.26), (4.28), (4.31) and (4.32)), hence no $l \in M^*_j$ can satisfy the above for more than two values of i_0, and it follows that $|\xi(l)| \leqq |\sigma(l)| + 2$. Moreover, from the above and the fact that both $\{{}^tM^*_{j(i)}\}_{t=1,2,3,\, i \in T^*}$ and $\{{}^tN_{j(i)}\}_{t=1,2,3,\, i \in T^*}$ are decompositions of M^*_j, it follows that if $l \in M_j^*$ for some $2 \leqq j \leqq m-1$, then actually $|\xi(l)| = |\sigma(l)| + 2$, since in this case (cf. (4.44.r), $1 \leqq r \leqq 4$) l must satisfy the above condition for two values of i_0. This proves (4.46).

We are now ready to prove (ar.3.2) for μ. By (4.46) and the induction hypothesis $|\xi(l)| \leqq |\sigma(l)| + 2 \leqq 2(n-1) + 2 = 2n$. We also wish to estimate a lower bound for the cardinality of the set $E = \{l : 1 \leqq l \leqq L : |\xi(l)| = 2n\}$; instead we shall estimate an upper bound for the cardinality of its complement E^c. By (4.46) we have

$$E^c \subset M_1^* \cup M_m^* \cup M_{m+1}^* \cup \left(\bigcup_{j=2}^{m-1} \bigcup_{i \in T^*} {}^3M_{j(i)}^* \right)$$
$$\cup \left\{ l \in \bigcup_{j=2}^{m-1} \bigcup_{i \in T^*} ({}^1M_{j(i)}^* \cup {}^2M_{j(i)}^*) : |\sigma(l)| < 2(n-1) \right\}.$$

This follows from the fact that for any other l, $|\sigma(l)| = 2n-2$, and $l \in M_j^*$ for some $2 \leqq j \leqq m-1$, and thus, by (4.46), $|\xi(l)| = 2n$. Recall the following estimates:

$$|M_1^*| = |M_m^*| = m^{n-1} \qquad \text{(by (4.20))},$$
$$|M_{m+1}^*| \leqq 2^{T^*|} L^{(n-1)/n} \qquad \text{(by (4.41) (the last line there) and (4.48))},$$
$$|{}^3M_{j(i)}^*| \leqq 3|T^*| 2^{|T^*|} cm^{n-2} \qquad \text{(by (4.37) and (4.34))},$$

and hence also

$$\left| \bigcup_{j=2}^{m-1} \bigcup_{i \in T^*} {}^3M_{j(i)}^* \right| < m|T^*| 3|T^*| 2^{|T^*|} cm^{n-2} = 3|T^*|^2 2^{|T^*|} cm^{n-1}.$$
$$|\{l : l \in {}^1M_{j(i)}^* \cup {}^2M_{j(i)}^*, |\sigma(l)| < 2(n-1)\}| \leqq 3cm^{n-2}.$$

(This follows from the induction hypothesis, and (ar.3.2) when applied to ${}^2\nu_{j(i)}$, ${}^1\nu_{j(i)}$ and ${}^1\tilde{\nu}_{j(i)}$. Note that the norm of each of these arrays is $\leqq m^{n-1}$.) Thus we also have

$$\left\{ l \in \bigcup_{j=2}^{m-1} \bigcup_{i \in T^*} ({}^1M_{j(i)}^* \cup {}^2M_{j(i)}^*) : |\sigma(l)| < 2(n-1) \right\} \Big|$$
$$< m \cdot |T^*| 3cm^{n-1} = 3|T^*| cm^{n-1}.$$

Adding all this together we obtain

$$|E^c| \leqq m^{n-1} + m^{n-1} + 2^{|T^*|} L^{(n-1)/n} + 3|T^*|^2 2^{|T^*|} cm^{n-1} + 3|T^*| cm^{n-1}$$
$$(4.47) \qquad < 9|T^*|^2 2^{|T^*|} cL^{(n-1)/n}$$
$$= c(n, |T^*|) L^{(n-1)/n}.$$

(Recall that $m^{n-1} \leqq L^{(n-1)/n}$.) This proves that μ satisfies (ar.3.2), and completes the proof that μ is an array of order n and constant $c(n, |T^*|)$, w.r.t. $\{\varphi_i\}_{i\in T^*}$. To conclude the proof of Theorem 8, we still have to show that μ is normal, i.e., that (ar.N) is satisfied too. So, let ξ be a subset of T^*, and we wish to estimate $|\Sigma_{l:\xi(l)=\xi}\,\varepsilon(l)|$. Note first that by (ar.3.2), if $|\xi| \neq 2n$ then

$$\left|\sum_{l:\xi(l)=\xi} \varepsilon(l)\right| \leqq c(n, |T^*|)L^{(n-1)/n}.$$

(Indeed, if $|\xi| > 2n$ then the sum is over the empty set, while if $|\xi| < 2n$ then the sum is over a set of cardinality $\leqq c(n, |T^*|)L^{(n-1)/n}$.) So, let $\xi \subset T^*$ with $|\xi| = 2n$ be given.

(4.48) ξ admits $\binom{2n}{2} < |T^*|^2$ representations of the form $\xi = \sigma \cup \{i_0\} \cup \{i_1\}$, with $|\sigma| = 2(n-1)$.

We also have

$$\sum_{l:\xi(l)=\xi} \varepsilon(l) = \sum_{\substack{\xi=\sigma\cup\{i_0\}\cup\{i_1\}\\ |\sigma|=2(n-1)}} \sum_{2\leqq j\leqq m-1} \sum_{\substack{l:\sigma(l)=\sigma\\ l\in L_{i_0}\cap L_{i_1}\\ l\in M_j^*}} \varepsilon(l). \tag{4.49}$$

Indeed, it follows from (4.44) and (4.44.r), $1 \leqq r \leqq 4$, and (4.46) (cf. the proof of (4.46)) that if $l \in M_1^* \cup M_m^* \cup M_{m+1}^*$ or $|\sigma(l)| < 2(n-1)$ then $|\xi(l)| \leqq 2n-1$. Let us examine the set

$$\{l : l \in M_j^*,\ l \in L_{i_0} \cap L_{i_1},\ \sigma(l) = \sigma\} \qquad (\text{where } |\sigma| = 2(n-1)).$$

Recall that in order to be an element of L_i (for $i \notin \sigma(l)$), $l \in M_j^*$ must satisfy either $l \in {}^1M_{j(i)}^* \cup {}^2M_{j(i)}^* \cup {}^3M_{j(i)}^* \cup {}^1N_{j(i)}$ or $l \in {}^2N_{j(i)} \cup {}^3N_{j(i)}$.

Note also that if $\sigma(l) = 2(n-1)$ then $l \notin {}^3N_{j(i)}^*$. Thus, l must satisfy the above with both $i = i_0$ and $i = i_1$, and since both the '$M_{j(i)}^*$'s and the '$N_{j(i)}$'s are disjoint for different values of i, we conclude that for $l \in M_j^*$ with $|\sigma(l)| = 2(n-1)$, l must be an element of both $({}^1M_{j(i_0)}^* \cup {}^2M_{j(i_0)}^*)$ and $({}^2N_{j(i_1)} \cup {}^3N_{j(i_1)})$. Recall that by (4.26)

$$N_{j(i_1)} = \tau^{-1}(i_1),$$

and for $l \in {}^1M_{j(i_0)}^* \cup {}^2M_{j(i_0)}^*$, $\tau(l) = \tilde{\tau}(\{i_0\} \cup \sigma(l))$. Thus, the conditions $l \in {}^1M_{j(i_0)}^* \cup {}^2M_{j(i_0)}^*$ and $\sigma(l) = \sigma$ actually determine the value of i_1, such that $l \in N_{j(i_1)}$. It follows that for some fixed j, σ, i_0 and i_1, the sum

$$\sum_{\substack{l:\sigma(l)=\sigma\\ l\in M_j^*\\ l\in L_{i_0}\cap L_{i_1}}} \varepsilon(l) = 0 \qquad \text{if } \tilde{\tau}(\{i_0\} \cup \sigma) \neq i_1,$$

and

$$\sum_{\substack{l:\sigma(l)=\sigma \\ l\in M_j^* \\ l\in L_{i_0}\cap L_{i_1}}} \varepsilon(l) = \sum_{\substack{l\in {}^1M^*_{j(i_0)}\cup {}^2M^*_{j(i_0)} \\ \sigma(l)=\sigma}} \qquad \text{if } \tilde{\tau}(\{i_0\}\cup\sigma) = i_1.$$

This last sum can be decomposed as

$$\sum_{\substack{l:\sigma(l)=\sigma \\ l\in {}^2M^*_{j(i_0)}}} \varepsilon(l) + \sum_{\substack{l:\sigma(l)=\sigma \\ x_l\in \operatorname{supp} {}^1\nu_{j(i_0)}}} \varepsilon(l) + \sum_{\substack{l:\sigma(l)=\sigma \\ x_l\in \operatorname{supp} {}^1\tilde{\nu}_{j(i_0)}}} \varepsilon(l).$$

Recall that ${}^2\nu_{j(i_0)}$, ${}^1\nu_{j(i_0)}$ and ${}^1\tilde{\nu}_{j(i_0)}$ are all normal arrays, of order $n-1$ and norm $\leqq m^{n-1}$, and thus, from an application of (ar.N) to these arrays, it follows that the modulus of each one of the last three sums does not exceed cm^{n-2}. Hence

$$\left| \sum_{\substack{l\in {}^1M^*_{j(i_0)}\cup {}^2M^*_{j(i_0)} \\ \sigma(l)=\sigma}} \varepsilon(l) \right| \quad \text{and also} \quad \left| \sum_{\substack{\sigma(l)=\sigma \\ l\in M_j^* \\ l\in L_{i_0}\cap L_{i_1}}} \varepsilon(l) \right|$$

are bounded by $3cm^{n-2}$. From the fact that for a given j there are at most $|T^*|$ $M_{j(i)}$'s, and from (4.49) (and (4.48)), it follows that

$$\begin{aligned} \left| \sum_{l:\xi(l)=\xi} \varepsilon(l) \right| &\leqq \binom{2n}{n} (m-2)|T^*|3cm^{n-2} \\ &\leqq |T^*|^2 m |T^*| 3cm^{n-2} \\ &= 3|T^*|^3 cm^{n-1} \\ &< c(n,|T^*|)L^{(n-1)/n}. \end{aligned}$$

This concludes the verification of (ar.N) for μ, and also the proof of Theorem 8.

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